§5.4 Koszul complex and Thom isomorphism

Let $\pi: V \to X$ be a G-equiv. vector bundle.
Let i be the zero-section.

<u>Goal</u> Provide a canonical resolution of $i_* \mathcal{O}_X$ by G-equiv. locally free sheaves.

<u>Def</u> Put $\lambda(V) := \sum_{i=0}^{\mathrm{rk}\, V} (-1)^i [\Lambda^i V] \in K^G(X)$.

The <u>Koszul complex</u> is

$$\cdots \to \Omega^2_{V/X} \to \Omega^1_{V/X} \to \Omega^0_{V/X} \to i_* \mathcal{O}_X.$$

Let $\varphi: V^\vee \to X$ be the dual vector bundle. Then

$$\Omega^j_{V/X} \simeq \pi^*(\Lambda^j V^\vee).$$

So the Koszul complex can be rewritten as

$$\cdots \to \pi^*(\Lambda^2 V^\vee) \to \pi^*(\Lambda^1 V^\vee) \to \mathcal{O}_V \to i_* \mathcal{O}_X.$$

<u>Prop</u> The Koszul complex is exact.
Therefore, $[i_* \mathcal{O}_X] = \pi^*(\lambda(V^\vee)) \in K^G(X)$.

Now we can use the Koszul resolution to define $i^*: K^G(V) \to K^G(X)$, by $[F] \mapsto \sum (-1)^i H^i(F \otimes \text{Koszul})$. Recall: $i^*[F] =$ alternating sum of cohomologies of $F \otimes$ (resolution of $i_* \mathcal{O}_X$)

<u>Lemma</u> $i^* \pi^*: K^G(X) \to K^G(X)$ is identity.
$i^* i_*: K^G(X) \to K^G(X)$ is $\lambda(V^\vee) \otimes -$.

Prop Let $i : N \hookrightarrow M$ be a G-equiv closed embedding of a smooth G-variety N into a smooth G-variety M.
Then $i^* i_* : K^G(N) \longrightarrow K^G(N)$ is $\lambda(T^*_N M) \otimes -$.

Cor If $0 \to V_1 \to V \to V_2 \to 0$ is a SES of G-equiv vector bundles, then $\lambda(V) = \lambda(V_1) \otimes \lambda(V_2)$.

Thm (Thom isomorphism)
Suppose $\pi : E \to X$ is an affine bundle.
Then $\pi^* : K^G_j(X) \xrightarrow{\sim} K^G_j(E)$ $\forall$ $j \geq 0$.

Cor Let $\pi : V \to X$ have zero-section $i : X \to V$.
Then i^* is the inverse to π^*, the Thom isomorphism.

2 Convolution and Thom isomorphism

The goal is to understand the compatibility between convolution and Thom isomorphism.

Setup

M_1, M_2 - smooth proj. G-varieties

$M_3 = pt$

$Z_{12} = M_1 \times M_2$

$Z_{23} = M_2 \times pt$

$Z_{13} := Z_{12} \circ Z_{23} = M_1 \times pt$

Convolution gives: $K^G(M_1 \times M_2) \otimes_{R(G)} K^G(M_2) \to K^G(M_1)$, i.e.,

$\rho_M : K^G(M_1 \times M_2) \longrightarrow \mathrm{Hom}_{R(G)}(K^G(M_2), K^G(M_1))$

Thom iso

$p_r : E_r \to M_r$ G-equiv. vector bundles

$i_r : M_r \hookrightarrow E_r$ zero-sections

$\bar{p} := id_{E_1} \times p_2 : E_1 \times E_2 \to E_1 \times M_2$

$\bar{i} := i_1 \times id_{M_2} : E_1 \times M_2 \hookrightarrow E_1 \times E_2$

$Z \subset E_1 \times E_2$ — closed G-stable subvariety

Assume $\bar{p}|_Z$ is proper.

Then $Z \circ E_2 \hookrightarrow E_1$ is well-defined.

So we have $K^G(Z) \otimes_{R(G)} K^G(E_2) \xrightarrow{\text{convolution}} K^G(Z \circ E_2) \xrightarrow{\text{pushforward}} K^G(E_1)$, i.e.

$\rho_E : K^G(Z) \longrightarrow \mathrm{Hom}_{R(G)}(K^G(E_2), K^G(E_1))$.

But the blue can be identified by Thom. How are $K^G(Z)$, $K^G(M_1 \times M_2)$ related?

Lemma The following diagrams (in K-theory and BM-homology) commute.

$$\begin{tikzcd} K^G(Z) \arrow[r, "\rho_E"] \arrow[d, "\bar{i}^* \circ \bar{p}_*"'] & \mathrm{Hom}_{R(G)}(K^G(E_2), K^G(E_1)) \arrow[d, equal, "\text{Thom}"] \\ K^G(M_1 \times M_2) \arrow[r, "\rho_M"] & \mathrm{Hom}_{R(G)}(K^G(M_2), K^G(M_1)) \end{tikzcd}.$$

$$\begin{tikzcd} H_*^{BM}(Z) \arrow[r, "\rho_E"] \arrow[d, "\bar{i}^* \circ \bar{p}_*"'] & \mathrm{Hom}(H_*^{BM}(E_2), H_*^{BM}(E_1)) \arrow[d, equal, "\text{Thom}"] \\ H_*^{BM}(M_1 \times M_2) \arrow[r, "\rho_M"] & \mathrm{Hom}(H_*^{BM}(M_2), H_*^{BM}(M_1)) \end{tikzcd}$$

Lemma Take $M_1 = M_2$, $E_1 = E_2$.
Assume $Z \circ Z \subset Z$. Then the following diagram commutes.

$$\begin{tikzcd} K^G(Z) \arrow[r] \arrow[d] & \mathrm{End}_{R(G)}(K^G(E)) \arrow[d, equal, "\text{Thom}"] \\ K^G(M \times M) \arrow[r] & \mathrm{End}_{R(G)}(K^G(M)) \end{tikzcd}$$