§5.1

Our goal is to understand G-equivariant sheaves.

Setup

G – linear algebraic group (i.e., closed subgroup scheme of GL_n)

X – G-variety, i.e. variety equipped with G-action

$G \times X \underset{p}{\overset{a}{\rightrightarrows}} X$ comes with two maps: a: action map (X as a G-variety) p: projection map

Motivation

Comes from G-invariant functions.

A G-invariant function $f: X \to \mathbb{C}$ is a function such that

$$f(g \cdot x) = f(x) \quad \forall x \in X, g \in G.$$

Another way to rewrite this: a^*f and p^*f are functions on $G \times X$,

$$a^*f(g,x) = f(g \cdot x) \quad \text{and} \quad p^*f(g,x) = f(x).$$

So f is G-invariant $\longleftrightarrow$ $a^*f = p^*f$.

There are also compatibility maps $G \times G \times X \underset{id \times a}{\overset{m \times id}{\rightrightarrows}} G \times X$,

coming from $f(g_1(g_2 x)) = f(x) = f((g_1 g_2)x)$, saying that

$$(m \times id)^* p^* f = (id \times a)^* a^* f.$$

Sheaf pullback

We work with $\mathcal{O}_X$-modules, so for a map $f: X \to Y$, and an $\mathcal{O}_Y$-module $\mathcal{F}$, define

$$f^*\mathcal{F} := \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{F}.$$

Def Let X be a G-variety, $\mathcal{F}$ an $\mathcal{O}_X$-module.
$\mathcal{F}$ is <u>G-equivariant</u> if

1) there is a <u>given</u> isomorphism $I: a^*\mathcal{F} \xrightarrow{\sim} p^*\mathcal{F}$,
2) the maps $G\times G\times X \longrightarrow X$ are related by

$$p_{23}^* I \circ (id_G \times a)^* I = (m \times id_X)^* I.$$

Remark
1) A similar definition works for arbitrary sheaves instead of $\mathcal{O}_X$-mod.
2) It's meaningless to ask <u>if</u> a sheaf is G-equivariant.
To be G-equivariant is to equip a sheaf with additional structure.

Ex For a G-variety X, then $\mathcal{O}_X$ is a G-equiv. sheaf by

$$a^*\mathcal{O}_X \simeq \mathcal{O}_{G\times X} \simeq p^*\mathcal{O}_X.$$

Ex Let $\mathcal{F}$ be a locally free sheaf on a G-variety X,
with total space $\pi: F \to X$.
Then: giving $\mathcal{F}$ a G-equiv. structure is equivalent to giving a G-action $\Phi: G\times F \to F$ on the total space, such that:

1) π commutes with G-action
(in particular, $g: F_x \longrightarrow F_{g\cdot x}$ on fibers)

2) on fibers, $\Phi(g,-)$ is a linear map of vector spaces.

2 Equivariant line bundles

Our main goal is to prove:

Main thm
G — linear alg. group
X — normal G-variety
$\mathcal{L}$ — line bundle on X

Then $\exists\ n \in \mathbb{Z}_{>0}$ such that $\mathcal{L}^{\otimes n}$ admits a (possibly not unique) G-equivariant structure.

We'll prove this through a series of lemmas. For convenience,

Def Say a variety X is "fairly affine" if it contains a Zariski-open dense subset of the form $\mathbb{C}^r \times (\mathbb{C}^\times)^s = \mathbb{A}^r \times \mathbb{G}_m^s$.

Lemma Let X be a normal variety.
1) If X is smooth then $\mathrm{Pic}(X) \simeq Cl(X) := \mathrm{Div}(X)/\mathrm{PDiv}(X)$.
2) For any X, the projection map $p: (\mathbb{C}^r \times (\mathbb{C}^\times)^s) \times X \to X$ induces an isomorphism via pullback $p^*: Cl(X) \xrightarrow{\sim} Cl(\mathbb{C}^r \times (\mathbb{C}^\times)^s \times X)$.
3) Say $U \subset X$ is open and $\{C_i\}$ are the irred. comps of $X \setminus U$ (of codim 1). Then we have an exact sequence
$$\mathbb{Z}\{C_i\} \longrightarrow Cl(X) \longrightarrow Cl(U) \longrightarrow 0.$$

Lemma Let M, X be smooth varieties with M "fairly affine" and projections $M \times X \xrightarrow{p_M} M$, $M \times X \xrightarrow{p_X} X$. Then any line bundle $\mathcal{L}$ on $M \times X$ is the tensor product of pullbacks p_M^*, p_X^* of line bundles on M, X. (To be explicit, $\mathcal{L} \simeq (p_M^* \mathcal{L}|_{M \times \{x\}}) \otimes (p_X^* \mathcal{L}|_{\{m\} \times X})$.)
Note: this still works for M, X normal and $m \in M$, $x \in X$ smooth points.

Proof
1) Set $U := \mathbb{C}^r \times (\mathbb{C}^\times)^s \subset M \Rightarrow M \setminus U$ has components $C_1, \dots, C_n$ of codim 1.
2) We have an exact seq. $\mathbb{Z}^n \to Cl(M \times X) \to Cl(U \times X) = Cl(X) \to 0$.
Here, $\mathbb{Z}^n = \mathbb{Z}\{[C_i \times X]\}$.
3) So $Cl(M \times X)$ is generated by $p_M^* D$ for $D \in \mathbb{Z}\{C_i\} \subset Cl(M)$ and $p^* E$ for $E \in Cl(X)$. $\square$

Lemma Let G be a linear alg. group. The G is "fairly affine."

Proof

1) The unipotent radical $G_u \subset G$ is $\simeq \mathbb{A}^t$, so it suffices to prove for $G' := G/G_u$, a reductive group.
2) Pick opposite Borels $B^+, B^- \subset G'$, with $T := B^+ \cap B^-$ a maximal torus.
3) $B^+ \cdot B^-$ is dense open, b/c $B^+ \cdot [e] \subseteq G/B^-$ is the unique dense open cell/orbit in the flag variety.
4) $B^+ \cdot B^- \simeq U^- \times T \times U^+ \simeq \mathbb{C}^r \times (\mathbb{C}^\times)^s$. $\square$

So now we can apply the lemma to $G \times X$.

Lemma $\mathrm{Pic}(G)$ is torsion.

Proof

1) G has finitely many connected comps, so by induction assume G is connected.
2) As varieties, $G \simeq \underbrace{G_u}_{\text{is } \mathbb{A}^t} \times \underbrace{(G/G_u)}_{\text{reductive}}$, so $Cl(G) \simeq Cl(G/G_u)$.

   Also everything is smooth so $Cl \simeq \mathrm{Pic}$. So assume G is reductive.
3) We have an exact seq. $\mathbb{Z}^n \to Cl(G) \to Cl(U^- \times T \times U^+) = 0$.

   free ab. grp by closures of Bruhat cells of codim 1

   simple reflections $s_\alpha \in W$, $\overline{B^+ s_\alpha B^-}$.
4) For G semisimple, $\overline{B^+ s_\alpha B^-} = V(f_\alpha)$ for some $f_\alpha \in \mathbb{C}[G]$, shown by BGG. So each cell has class $0 \in Cl(G) \Rightarrow Cl(G) = 0$.

   principal divisor

   In general, find a finite cover $G' \to G$, $G' \simeq (\text{semisimple}) \times (\text{torus})$, then $\deg(G' \to G)$ kills the divisor. $\square$

Sketch of proof of main thm

1) Let $\mathcal{L}$ be a line bundle on X. Then $a^*\mathcal{L} \simeq (p_G^* E) \otimes (p_X^* F)$.

   In particular, picking $e \in G$, $F \simeq a^*\mathcal{L}|_{\{e\} \times X} \simeq \mathcal{L}$, so $a^*\mathcal{L} \simeq (p_G^* E) \otimes (p_X^* \mathcal{L})$
2) E is line bundle on G, so is torsion; so $E^{\otimes n} \simeq \mathcal{O}_G \Rightarrow a^*\mathcal{L}^{\otimes n} \simeq p_X^* \mathcal{L}^{\otimes n}$.
3) So $a^*(\mathcal{L}^{\otimes n}) \xrightarrow{\sim} p^*(\mathcal{L}^{\otimes n})$. Just need to check compatibilities with $G \times G \times X \to G \times X$. $\square$

3 More on G-equivariant sheaves

Cor (of main thm)

Let X be a smooth quasi-projective G-variety.
Then $\exists$ an ample G-equivariant line bundle on X.

Proof
The embedding $\pi : X \hookrightarrow \mathbb{P}^n$ is induced by $\pi^*\mathcal{O}(1)$, is ample.
The main thm says that some tensor power of it is G-equiv. $\square$

Algebraic actions

Let M be a (possibly ∞-dim) G-module.

Def
The G-action is <u>algebraic</u> if $\forall m \in M$, $\exists$ a finite-dim G-stable subspace $V_m \subset M$ containing m.

The way to think about it is as follows. $\mathbb{C}[G] := \mathcal{O}_G(G)$ is a <u>coalgebra</u>. Then

M is an algebraic G-module	$\longleftrightarrow$	$\forall m \in M$, $\exists$ a finite set $\{m_i\}$ s.t. $g \cdot m \in \sum \mathbb{C} m_i \subset M$	$\longleftrightarrow$	we get a map $M \to \mathbb{C}[G] \otimes M$, $m \mapsto \sum f_i \otimes m_i$ s.t. $g \cdot m = \sum f_i(g) \cdot m_i$ $\forall g \in G$	$\longleftrightarrow$	M is a $\mathbb{C}[G]$-comodule

Remark
Being a comodule implicitly has some finiteness assumptions since the tensor $\mathbb{C}[G] \otimes M$ consists of <u>finite</u> sums.
In general, a G-module M just gives a map $M \to \mathrm{Maps}(G, M)$.
$$\mathrm{Maps}(G,M) \supset \mathbb{C}[G] \otimes M.$$

Lemma
If $\mathcal{F}$ is a G-equiv. coherent sheaf on a G-variety X, then $\Gamma(X, \mathcal{F})$ is an algebraic G-module.

Proof
An algebraic G-module is just a $\mathbb{C}[G]$-comodule. That's given by
$$\Gamma(X, \mathcal{F}) \xrightarrow{a^*} \Gamma(G \times X, a^*\mathcal{F}) \xrightarrow{\sim} \Gamma(G \times X, p^*\mathcal{F}) = \mathbb{C}[G] \otimes \Gamma(X, \mathcal{F}).$$
$\square$

Thm (Equivariant projective embedding)

Let X be a normal quasi-projective G-variety. We already know $X \hookrightarrow \mathbb{P}(V)$.
But $\exists$ (f.dim) V, $\rho: G \to GL(V)$ so V is a G-rep. and an equivariant embedding $i: X \hookrightarrow \mathbb{P}(V)$, i.e. $\rho(g)\cdot i(x) = i(g\cdot x)$.

Proof

1) Since X is quasi-proj., we have $X \hookrightarrow \underset{\text{projective}}{\overline{X}} \xhookrightarrow{\pi} \mathbb{P}(\Gamma(\overline{X},\mathcal{L})^*)$ for some ample line bundle $\mathcal{L}$ on $\overline{X}$, and $\mathcal{L} = \pi^*\mathcal{O}(1)$.

2) $(\mathcal{L}|_X)^{\otimes r}$ is G-equiv, so replace $\mathcal{L}$ by $\mathcal{L}^{\otimes r}$.

3) We have $\underset{\substack{\text{not } G\text{-stable,}\\ \text{finite-dim}}}{\Gamma(\overline{X},\mathcal{L})} \subset \Gamma(X,\mathcal{L})$. ← G-stable, possibly ∞-dim

But $\Gamma(X,\mathcal{L})$ is an algebraic G-module, so $\exists$ finite dim $\Gamma(\overline{X},\mathcal{L}) \subset \underset{G\text{-stable}}{V} \subset \Gamma(X,\mathcal{L})$.

4) Since $\Gamma(\overline{X},\mathcal{L})$ already separates pts and tangent vectors of $\overline{X}$, hence X, it follows that V does as well. So $X \hookrightarrow \mathbb{P}(V^*)$ is G-equiv. embedding. $\square$

To take resolutions by locally free <u>G-equiv</u> sheaves, we need to know surjections exist.

Prop Let X be a smooth (or just normal) quasiprojective G-variety. Then <u>any</u> G-equiv. coherent sheaf on X is surjected onto by a G-equiv. locally free sheaf.

Proof

1) $X \hookrightarrow \overline{X}$ ← projective. Extend $\mathcal{F}$ to $\overline{\mathcal{F}}$ on $\overline{X}$; this may not be G-equiv. Fix $\mathcal{L}$ ample on $\overline{X}$.

2) Pick n large; then:
   1) $\mathcal{L}^{\otimes n}|_X$ is G-equiv,
   2) $\overline{\mathcal{F}} \otimes \mathcal{L}^{\otimes n}$ is globally generated,
   3) $\mathcal{F} \otimes \mathcal{L}^{\otimes n}|_X$ is generated by $\Gamma(\overline{X}, \overline{\mathcal{F}} \otimes \mathcal{L}^{\otimes n})$.

3) We have $\underset{\text{finite-dim.}}{\Gamma(\overline{X}, \overline{\mathcal{F}} \otimes \mathcal{L}^{\otimes n})} \subset \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}|_X)$. ← G-stable, <u>algebraic</u> G-module

Since $\Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}|_X)$ is an <u>algebraic</u> G-module, $\exists$ a finite-dim G-stable subspace $\Gamma(\overline{X}, \overline{\mathcal{F}} \otimes \mathcal{L}^{\otimes n}) \subset V \subset \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}|_X)$ which generates $\Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}|_X)$.

4) So $V \otimes \mathcal{O}_X \twoheadrightarrow \mathcal{F} \otimes \mathcal{L}^{\otimes n}|_X \Rightarrow V \otimes \mathcal{L}^{\otimes -n}|_X \twoheadrightarrow \mathcal{F}$. $\square$

Now we need to know that everything extends from open dense subsets in the G-equivariant setting.

Prop Let $\mathcal{F}$ be a G-equivariant coherent sheaf on X. Let X be an open dense subset in a projective G-variety $\overline{X}$.

1) There's a G-equiv coherent sheaf $\overline{\mathcal{F}}$ on $\overline{X}$ so that $\overline{\mathcal{F}}|_X = \mathcal{F}$.
2) Let $f: \mathcal{F} \to \mathcal{G}$ be a G-equiv morphism of G-equiv sheaves on X. Then there's a G-equiv morphism of G-equiv sheaves $\overline{f}: \overline{\mathcal{F}} \to \overline{\mathcal{G}}$ on $\overline{X}$ extending f.

Proof

1) $\pi: V \otimes \mathcal{L}^\vee|_X^{\otimes n} \twoheadrightarrow \mathcal{F}$.
2) $W \otimes \mathcal{L}^\vee|_X^{\otimes m} \twoheadrightarrow \ker(\pi)$.
3) $\mathcal{F} = \operatorname{coker}\left(W \otimes \mathcal{L}^\vee|_X^{\otimes m} \xrightarrow{f} V \otimes \mathcal{L}^\vee|_X^{\otimes n}\right)$
4) $f \longleftrightarrow$ G-invariant section $s \in \operatorname{Hom}(W, V) \otimes \mathcal{L}^{\otimes(m-n)}$.
5) Form divisor $D \subset \overline{X}$ of all $x \in \overline{X}$ where $\operatorname{Hom}(W,V)$ has a pole. Then s is a G-invariant section $\operatorname{Hom}(W,V) \otimes \mathcal{L}^{\otimes(m-n)}(k \cdot D)$ for $k \gg 0$.
6) Then $s \longleftrightarrow \overline{f}: W \otimes \mathcal{L}^{\otimes -m} \to V \otimes \mathcal{L}^{\otimes -n}(k \cdot D)$, G-equiv. morphism. Then $\overline{\mathcal{F}} := \operatorname{coker} \overline{f}$ works.
7) Part 2 is very similar. $\square$

Prop If X is a smooth quasi-projective G-variety, then any G-equivariant coherent sheaf has a finite resolution by locally free G-equivariant sheaves.

Proof We can definitely take a resolution by repeatedly surjecting onto the kernels: $\cdots \to \mathcal{F}_4 \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1 \twoheadrightarrow \mathcal{F}$.
To see that it terminates, it suffices to use Hilbert syzygy thm (forget the G-equiv. structure), which says that $\mathcal{F}_{\dim X+1} = 0$ $\square$