§ 4.4 Stabilization

This will be a shorter summary, skipping many technical details.

Our goal is to understand the behavior of the constructions in §4.2 as $d \to \infty$.

Setup

Fix $r \in \{0, 1, \dots, n-1\}$. Our $d \in \mathbb{Z}_{>0}$ will always be $d = r + k \cdot n$, so any two d's are equal mod n.

Recall constructions in §4.2: we have N_d, M_d, Z_d, F_d.

Fix $e := \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \in \mathfrak{gl}_n$ to be the nilpotent Jordan block of size n.

Define the embedding $i: \mathfrak{gl}_d \hookrightarrow \mathfrak{gl}_{d+n}$, $x \longmapsto x \oplus e = \left(\begin{array}{c|c} x & \\ \hline & e \end{array} \right)$.

Goal: Construct $N_\infty, M_\infty, Z_\infty$, etc.

2/ Constructing N_∞

Fix the remainder $r \in \{0, 1, \dots, n-1\}$.

Then define $\mathbb{C}^{r+\infty} := \mathbb{C}^r \times \prod_{j\geq 0} \mathbb{C}^n$.

Let $\Gamma^k := \prod_{j\geq k} \mathbb{C}^n$. Then we have SES

$$0 \to \Gamma^k \to \mathbb{C}^{r+\infty} \to \mathbb{C}^{r+k\cdot n} \to 0.$$

Then we construct

$$GL_{r+\infty} := \{ g \in GL(\mathbb{C}^{r+\infty}) \mid g|_{\Gamma^k} = \mathrm{Id} \text{ for } k \gg 0 \}.$$

Note that $GL_{r+k\cdot n} \subset GL_{r+\infty}$ via $\{ g \in GL(\mathbb{C}^{r+\infty}) \mid g|_{\Gamma^k} = \mathrm{Id} \}$,

and $GL_{r+\infty} = \varinjlim_k GL_{r+k\cdot n}$.

So all elements should be "eventually Id".

So set $e_{\Gamma^k} := \prod_{j\geq k} e$.

Similarly,

$$N_{r+\infty} := \{ x \in \mathrm{End}(\mathbb{C}^{r+\infty}) \mid x^n = 0,\ x|_{\Gamma^k} = e_{\Gamma^k} \text{ for } k \gg 0 \}.$$

<u>Note</u>: $N_{r+\infty}$ is <u>not</u> a cone variety!

Similarly, $N_{r+\infty}$ can be realized as a direct limit:

we have $N_r \overset{i}{\hookrightarrow} N_{r+n} \overset{i}{\hookrightarrow} N_{r+2n} \overset{i}{\hookrightarrow} \cdots$

$$\Rightarrow \quad N_{r+\infty} = \varinjlim_k N_{r+k\cdot n}.$$

Similar to the finite-dim case, $GL_{r+\infty} \curvearrowright N_{r+\infty}$ by conjugation.

Remark: The $GL_{r+\infty}$-orbits are naturally in bijection with things called "Dirac seas."

3 Dependence on d

Fix $\mathcal{F} = (0 \subseteq \mathbb{C} \subseteq \mathbb{C}^2 \subseteq \mathbb{C}^3 \subseteq \cdots \subseteq \mathbb{C}^n)$ the standard full flag in $\mathbb{C}^n$, the unique full flag fixed by $e \in \mathfrak{gl}_n$.

1) We have a system

$$\mathcal{F}_d \xhookrightarrow{i} \mathcal{F}_{d+n} \xhookrightarrow{i} \mathcal{F}_{d+2n} \xhookrightarrow{i} \cdots$$

where $i(F) = F \oplus \mathcal{F} := (0 \oplus 0 \subseteq F_1 \oplus \mathbb{C} \subseteq F_2 \oplus \mathbb{C}^2 \subseteq \cdots \subseteq \mathbb{C}^d \oplus \mathbb{C}^n)$

2) We have maps $\mu : M_d \to N_d$, $(x, F) \mapsto x$.
These fit into a commutative diagram / system

$$\begin{tikzcd}
M_d \arrow[r, hook, "i"] \arrow[d, "\mu"] & M_{d+n} \arrow[r, hook, "i"] \arrow[d, "\mu"] & M_{d+2n} \arrow[r, hook, "i"] \arrow[d, "\mu"] & \cdots \\
N_d \arrow[r, hook, "i"] & N_{d+n} \arrow[r, hook, "i"] & N_{d+2n} \arrow[r, hook, "i"] & \cdots
\end{tikzcd}$$

3) Lemma $\forall x \in N_d$, we have $i(\mathcal{F}_x) = \mathcal{F}_{i(x)}$.

Cor

$$i^* : H(\mathcal{F}_{i(x)}) \xrightarrow{\sim} H(\mathcal{F}_x).$$

4) Since $Z_d := M_d \times_{N_d} M_d$, we have maps

$$Z_d \xhookrightarrow{i} Z_{d+n} \xhookrightarrow{i} Z_{d+2n} \xhookrightarrow{i} \cdots$$

We have the comm. diagram:

$$\begin{tikzcd} Z_d \arrow[r, hook, "i"] \arrow[d, hook] & Z_{d+n} \arrow[d, hook] \\ M_d \times M_d \arrow[r, hook, "i \times i"] & M_{d+n} \times M_{d+n} \end{tikzcd}.$$

This induces restriction with support map:

$$i^*: H_\bullet^{BM}(Z_{d+n}) \longrightarrow H_\bullet^{BM}(Z_d)$$
$$c \longmapsto c \cap [M_d \times M_d].$$

Furthermore, it restricts to a map

$$i^*: H(Z_{d+n}) \longrightarrow H(Z_d).$$

Main thm 1

1) $i^*: H(Z_{d+n}) \to H(Z_d)$ is a hom. of algebras wrt convolution.

2) The $H(Z_d) \otimes H(\mathcal{F}_x) \to H(\mathcal{F}_x)$ fit into the system:

$$\begin{tikzcd} \cdots \arrow[r] & H(Z_{d+2n}) \otimes H(\mathcal{F}_{i^2(x)}) \arrow[r, "i^* \otimes i^*"] \arrow[d] & H(Z_{d+n}) \otimes H(\mathcal{F}_{i(x)}) \arrow[r, "i^* \otimes i^*"] \arrow[d] & H(Z_d) \otimes H(\mathcal{F}_x) \arrow[d] \\ \cdots \arrow[r] & H(\mathcal{F}_{i^2(x)}) \arrow[r, "i^*", "\sim"'] & H(\mathcal{F}_{i(x)}) \arrow[r, "i^*", "\sim"'] & H(\mathcal{F}_x) \end{tikzcd}$$

5 Profinite completion

$\mathcal{U}(\mathfrak{sl}_n)$ is an infinite-dimensional algebra.
We'll now take the profinite completion:

Remark
The topology is separating.

$$\hat{U} := \varprojlim \mathcal{U}(\mathfrak{sl}_n)/I$$

the inverse limit over all finite-dim quotients,
equivalently, over all (two-sided) ideals I of finite codimension.

Since $Z(SL_n) = \mu_n$, the roots of unity, we have a weight space decomp for any f.d. SL_n-module M:

$$M = \bigoplus_{\chi \in (\mu_n)^\vee} M_\chi$$

where $(\mu_n)^\vee$ is the Cartier dual (= Pontryagin dual) of μ_n; this is $\underline{\mathbb{Z}/n\mathbb{Z}}$, and M_χ is the subspace for which $Z(SL_n)$ acts by χ.

In particular, $\mathcal{U}(\mathfrak{sl}_n)/I = \bigoplus_{r \in \mathbb{Z}/n\mathbb{Z}} (\mathcal{U}(\mathfrak{sl}_n)/I)_r$.

Therefore $\hat{U} = \varprojlim \mathcal{U}(\mathfrak{sl}_n)/I = \bigoplus_{r \in \mathbb{Z}/n\mathbb{Z}} \varprojlim (\mathcal{U}(\mathfrak{sl}_n)/I)_r = \bigoplus_{r \in \mathbb{Z}/n\mathbb{Z}} \hat{U}_r$.

$\hat{U}_r \subset \hat{U}$ is the subalgebra characterized by the property that: for any (rational) simple SL_n-module with central character χ_d, if $d \equiv r \pmod n$ then $\hat{U}_r$ acts nontrivially;
if $d \not\equiv r \pmod n$ then $\hat{U}_r$ acts trivially.

The maps $\phi_d : \mathcal{U}(\mathfrak{sl}_n) \twoheadrightarrow H(Z_d)$ with kernel

$$I_d := \mathrm{Ann}((\mathbb{C}^n)^{\otimes d})$$

are compatible with i^*:

$$\begin{tikzcd}
 & \mathcal{U}(\mathfrak{sl}_n) \arrow[dl, two heads, "\phi_d"'] \arrow[d, two heads, "\phi_{d+n}"] \arrow[dr, two heads, "\phi_{d+2n}"] & & \\
H(Z_d) & H(Z_{d+n}) \arrow[l, two heads, "i^*"'] & H(Z_{d+2n}) \arrow[l, two heads, "i^*"'] & \cdots \arrow[l, two heads]
\end{tikzcd}$$

This gives us a map $\mathcal{U}(\mathfrak{sl}_n) \longrightarrow \varprojlim_k H(Z_{r+k\cdot n})$.

Main thm 2

For each $r \in \{0, 1, \ldots, n-1\}$, we have an iso of complete topological algebras $\hat{U}_r \simeq \varprojlim_k H(Z_{r+k\cdot n})$.

This realizes the natural map $\mathcal{U}(\mathfrak{sl}_n) \to \varprojlim_k H(Z_{r+k\cdot n})$ as $\mathcal{U}(\mathfrak{sl}_n) \hookrightarrow \hat{U} \twoheadrightarrow \hat{U}_r \overset{\sim}{=} \varprojlim_k H(Z_{r+k\cdot n})$.

Proof Recall that $H(Z_d) = \mathcal{U}(\mathfrak{sl}_n) / I_d$, $I_d = \mathrm{Ann}((\mathbb{C}^n)^{\otimes d})$.
But the center acts on $\mathbb{C}^n$ by χ_1, hence acts on $(\mathbb{C}^n)^{\otimes d}$ by $\chi_d = \chi_r$ since $d = r+k\cdot n$. So

$$\varprojlim H(Z_{r+k\cdot n}) \hookrightarrow \hat{U}_r.$$

But every irrep with central character χ_r appears in $(\mathbb{C}^n)^{\otimes r+k\cdot n}$ for $k \gg 0$, so actually $\varprojlim H(Z_{r+k\cdot n}) \xrightarrow{\sim} \hat{U}_r$.

$\square$

6 Infinite-dimensional interpretation

In the f.d. setting,

$$M_d = T^* \mathcal{F}_d,$$

$$\begin{array}{ccc} M_d & \ni & (x,F) \\ \mu\downarrow & & \downarrow \\ N_d & \ni & x \end{array}$$

And $Z_d \circ \mathcal{F}_x = \mathcal{F}_x \Rightarrow H(Z_d) \curvearrowright H(\mathcal{F}_x)$.

In the ∞-dim setting:

- We constructed $N_{r+\infty}$
- Define

$$\mathcal{F}_{r+\infty} = \{ F = (0 \subset F_1 \subset \cdots F_n \subset \mathbb{C}^{r+\infty}) \mid F \cap \Gamma^k = \mathcal{F}(\Gamma^k) \text{ for } k \gg 0 \}$$

$$= \varinjlim \mathcal{F}_{r+k\cdot n} \quad \text{under maps}$$

$$\mathcal{F}_r \xhookrightarrow{i} \mathcal{F}_{r+n} \xhookrightarrow{i} \mathcal{F}_{r+2n} \xhookrightarrow{i} \cdots$$

- Define

$$M_{r+\infty} = \{ (x,F) \in N_{r+\infty} \times \mathcal{F}_{r+\infty} \mid x(F_i) \subseteq F_{i-1} \ \forall i \}$$

$$= \varinjlim M_{r+k\cdot n} \quad \text{under maps}$$

$$M_r \xhookrightarrow{i} M_{r+n} \xhookrightarrow{i} M_{r+2n} \xhookrightarrow{i} \cdots$$

This plays the role of the cotangent bundle to $\mathcal{F}_{r+\infty}$ (although there's no zero section).

- We again have a map $\mu: M_{r+\infty} \longrightarrow N_{r+\infty}$,

$$(x,F) \longmapsto x$$

the fibers are <u>finite-dimensional</u> <u>projective varieties</u>.

- Define $Z_{r+\infty} := M_{r+\infty} \times_{N_{r+\infty}} M_{r+\infty} = \varinjlim Z_{r+k\cdot n}$ (under i maps).
- Then $\hat{U}_r = \varprojlim_k H(Z_{r+k\cdot n})$ " $= H(Z_{r+\infty})$ " <u>can be viewed as</u> the algebra of infinite sums of irred. comps of $Z_{r+\infty}$.