§3.7

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Main thm (Jacobson-Morozov)

Let $\mathfrak{g}$ be semisimple Lie alg$/k$ $\leftarrow$ field of char 0. For all nilpotent $e \in \mathfrak{g}$, there exists $h, f \in \mathfrak{g}$ s.t. (e,f,h) form an $\mathfrak{sl}_2$-triple:

1) $[h,e]=2e$, $[h,f]=-2f$, $[e,f]=h$
2) $\exists$ Lie alg hom $\gamma : \mathfrak{gl}_2(\mathbb{C}) \to \mathfrak{g}$ sending $e \mapsto e$, $h \mapsto h$, $f \mapsto f$
3) $h \in \mathfrak{g}^{ss}$ and $f \in \mathcal{N}$.

Proof Postponed to end of §3.7. $\square$

Cor Given $e \in \mathcal{N} \subset \mathfrak{g}$, $\exists$ map $\gamma : SL_2(\mathbb{C}) \to G$ s.t. $d\gamma : \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto e$.

Def We say (e,f,h) is an $\mathfrak{sl}_2$-triple.

Cor Since $\mathfrak{sl}_2 \ni h \leftrightarrow \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \in SL_2(\mathbb{C})$, we get a hom. $\gamma : \mathbb{C}^\times \to G$ s.t. $\gamma(t) e \gamma(t)^{-1} = t^2 e$.

Rmk Note that this triple is <u>not</u> unique!! In fact, non-uniqueness is measured by:

Prop Fix $e \in \mathcal{N} \subset \mathfrak{g}$. Then $\gamma : \mathfrak{sl}_2 \to \mathfrak{g}$ sending $e \mapsto e$ is determined uniquely up to conjugation by the unipotent radical of $Z_G(e)$, i.e., the $\mathfrak{u} \subset Z_{\mathfrak{g}}(e)$ where $\mathfrak{u} = \mathrm{im}(e) \cap \underset{Z_{\mathfrak{g}}(e)}{\overset{\shortparallel}{\ker(e)}}$.

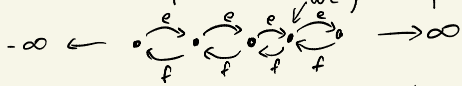
Proof Again postponed to end of §3.7. $\square$

Ex Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. Then any $e \in \mathcal{N}$ is conjugate to a direct sum of Jordan blocks, so it suffices to deal with a single Jordan block. But we can write down h and f explicitly:

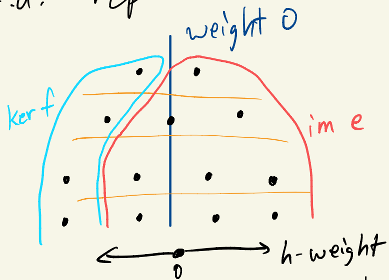
$$h = \begin{pmatrix} m-1 & 0 & & & \\ 0 & m-3 & 0 & & \\ & & \ddots & & \\ & & & & \\ 0 & \dots & & 0 & -m+1 \end{pmatrix}, \qquad e = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & 0 & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix},$$

$$f = \begin{pmatrix} 0 & 0 & & & \\ m-1 & 0 & 0 & & \\ 0 & 2(m-2) & 0 & & \\ & & \ddots & & \\ & & -2(m-2) & 0 & 0 \\ & & & -(m-1) & 0 \end{pmatrix}.$$

Recall that every f.d. $\mathfrak{sl}_2$-rep is a direct sum of irreps,
and every f.d. irrep looks like: weight spaces of h, integer weights



Therefore, every f.d. rep looks like:



rows are the $\oplus$-decomp into irreps.

e : increases weight by 2

f : decreases weight by 2

This diagram is symmetric about $h=0$.

Cor Assume V is a f.d. $\mathfrak{sl}_2$-rep.
If $v \in V$ with $f \cdot v = 0$ and $h \cdot v = -m \cdot v$, then $m \in \mathbb{Z}_{\geq 0}$ and $e^{m+1} \cdot v = 0$.

Cor If $e \in \mathfrak{g}$ is nilpotent, then it acts nilpotently on any f.d. $\mathfrak{g}$-module.

Proof Use the fact that e can be extended to an $\mathfrak{sl}_2$-triple. $\square$

1) Fix $e \in \mathcal{N} \subset \mathfrak{g}$.

2) Choose an $\mathfrak{sl}_2$-triple (e,f,h), so that $\mathfrak{sl}_2 \hookrightarrow \mathfrak{g}$, and $\mathfrak{g}$ is an $\mathfrak{sl}_2$-module

3) $\mathrm{ad}\, h$ acts on $\mathfrak{g}$ by $\mathbb{Z}$-weights. This induces a grading

$$\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n, \qquad \mathfrak{g}_n := \{x \in \mathfrak{g},\ \mathrm{ad}\, h\,(x) = n \cdot x\}.$$

Note that $\mathrm{Lie}\, Z_G(e) = Z_{\mathfrak{g}}(e) = \ker(\mathrm{ad}\, e)$.

Cor

1) All eig's of $\mathrm{ad}\, h$ on $Z_{\mathfrak{g}}(e)$ are in $\mathbb{Z}_{\geq 0}$.

2) If all eig's of $\mathrm{ad}\, h$ are even, then $\dim Z_{\mathfrak{g}}(e) = \dim Z_{\mathfrak{g}}(h)$.

Proof

Use the picture.

1) $Z_{\mathfrak{g}}(e)$ is all of the highest weight spaces in each row.
By symmetry, all weights are ≥ 0.

2) The rows are in bijection with irreps in the direct sum decomp.
$\dim Z_G(h) = \dim \mathfrak{g}_0 = \#$ irreps, and $\dim Z_{\mathfrak{g}}(e) = \#$ irreps. $\square$

since even weights $\Rightarrow$ each irrep has exactly one 0-weight space.

Part (2) of this Corollary gives us an easy proof of:

Thm Choose a decomp $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Let $\{e_i\} \subset \mathfrak{n}_+$ be root vectors for simple roots. Then $\sum e_i$ is regular nilpotent.

Proof Set $e = \sum e_i$; we need to construct (e,f,h) s.t. eig's of h are even, so that the Cor implies $\dim Z_{\mathfrak{g}}(e) = \dim Z_{\mathfrak{g}}(h) = \dim \mathfrak{h} \Rightarrow e$ is maximal.

1) Fix simple roots α_i. We are given $e_i \in \mathfrak{n}_{\alpha_i}$; also fix $e_{-i} \in \mathfrak{n}_{-\alpha_i}$.

2) Fix $h_i = c_i [e_i, e_{-i}]$ s.t. $\alpha_i(h_i) = 2$. These form a basis of $\mathfrak{h}$.

3) Fix $h = \sum \mu_i h_i$ s.t. $\alpha_i(h) = 2\ \forall i$.

4) To complete the $\mathfrak{sl}_2$-triple, set $f := \sum \mu_i e_{-i}$.

Now by construction, (e,f,h) is an $\mathfrak{sl}_2$-triple (this can be verified by direct computation of the commutator relations), and $\mathrm{ad}\, h$ has only even eig's (since $\alpha_i(h) = 2\ \forall$ simple roots). $\square$

4 Slices

We are interested in <u>Slodowy slices</u>.

Def Fix $e \in \mathcal{N} \subset \mathfrak{g}$, and choose an $\mathfrak{sl}_2$-triple (e,f,h).
Let $\mathbb{O} = G.e$ be the orbit of e and $\mathfrak{s} := Z_{\mathfrak{g}}(f)$.
The <u>Slodowy slice</u>, aka <u>standard slice</u>, is $e+\mathfrak{s} = e + Z_{\mathfrak{g}}(f)$.

Prop The affine space $e+\mathfrak{s}$ is transverse to $\mathbb{O}$ in $\mathfrak{g}$, and $(e+\mathfrak{s}) \cap \mathbb{O} = e$.

Proof
1) To show transversality, we need to show $T_e(e+\mathfrak{s}) \oplus T_e(\mathbb{O}) \simeq T_e(\mathfrak{g})$.
But $T_e(e+\mathfrak{s}) = \mathfrak{s}$, $T_e\mathbb{O} = T_e(G.e) = [\mathfrak{g}, e] = \mathrm{im}(\mathrm{ad}\, e)$, and $T_e\mathfrak{g} = \mathfrak{g} = \ker(\mathrm{ad}\, f) \oplus \mathrm{im}(\mathrm{ad}\, e) = T_e(e+\mathfrak{s}) \oplus T_e(\mathbb{O})$.

2) To show that $(e+\mathfrak{s}) \cap \mathbb{O} = e$, we define a $\mathbb{C}^\times$-action on $e+\mathfrak{s}$.
Recall the map $\gamma : \mathbb{C}^\times \to G$, s.t. $e = t^2 \cdot Ad(\gamma(t)^{-1})(e)$.
Then define $\mathbb{C}^\times \times (e+\mathfrak{s}) \to e+\mathfrak{s}$ by $(t, e+s) \mapsto e + t^2 \cdot Ad(\gamma(t)^{-1})(s) = t^2 \cdot Ad(\gamma(t^{-1}))(e+s)$.
Now, $\mathfrak{s} = \ker(\mathrm{ad}\, f)$ has only <u>lowest</u>, hence <u>non-positive</u>, eig's for ad h action.
So $Ad(\gamma(t)^{-1})$ acts by <u>nonnegative</u> weights; $t \mapsto t^2 \cdot Ad(\gamma(t)^{-1})$ acts by <u>strictly positive</u> weights, so $\forall s \in \mathfrak{s}$, we have a contracting map $\lim_{t \to 0} t.(e+s) = e$.
It remains to know that $\mathbb{O} \cap (e+\mathfrak{s})$ is $\mathbb{C}^\times$-stable, see Prop 3.7.6. □

Cor Lemma 3.2.20 tells us that for some open nbhd $e \in U \subset \mathfrak{g}$,
$\mathcal{N} \cap (e+\mathfrak{s}) \cap U$ is a transverse slice to $\mathbb{O}$ <u>in $\mathcal{N}$</u>.

Prop

$$\begin{tikzcd} \mu^{-1}(e) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mu^{-1}(e+\mathfrak{s}) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{\mathcal{N}} \arrow[d, "\mu"] \\ \{e\} \arrow[r, hook] & e+\mathfrak{s} \arrow[r, hook] & \mathcal{N} \end{tikzcd}$$

$\mu^{-1}(e)$ is a homotopy retract of $\mu^{-1}(e+\mathfrak{s})$.

Proof Omitted, see book. □

5 Structure of $Z_G(e)$.

0) Clearly $\operatorname{Lie} Z_G(e) = Z_{\mathfrak{g}}(e)$.

1) Pick $\mathfrak{sl}_2$-triple (e,f,h). Then ad h induces grading $\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$.
Put $Z_n := Z_{\mathfrak{g}}(e) \cap \mathfrak{g}_n$. This gives a grading $Z_{\mathfrak{g}}(e) = \bigoplus_{n\geq 0} Z_n$.

2) This sum runs over only nonneg. integers b/c $Z_{\mathfrak{g}}(e) = \ker(\operatorname{ad} e)$, only highest weights.

3) Therefore $\mathfrak{u} := \bigoplus_{n>0} Z_n$ is a nilpotent ideal in $Z_{\mathfrak{g}}(e)$, corresponding to unipotent subgroup $U \subset Z_G(e)$.

Lemma

1) $\mathfrak{u} = \operatorname{im}(e) \cap \ker(e)$.

2) $h + \mathfrak{u} = U\cdot h$ is stable under adjoint U-action, and is a single U-orbit.

Proof

1) Immediate by structure diagram of $\mathfrak{sl}_2$-modules.

2) It's clear that $[h,\mathfrak{u}] = \mathfrak{u}$, so $T_h(U\cdot h) = T_h(h+\mathfrak{u})$. This implies that $U\cdot h$ is dense + closed in $h+\mathfrak{u}$, hence all of $h+\mathfrak{u}$. □
↳ U is unipotent

Now we can prove two results from the beginning of §3.7.

Prop Fix $e \in \mathcal{N} \subset \mathfrak{g}$.
Then $\gamma : \mathfrak{sl}_2 \to \mathfrak{g}$ sending $e \mapsto e$ is determined uniquely up to conjugation by the unipotent radical $U \subset Z_G(e)$, i.e., the $\mathfrak{u} \subset Z_{\mathfrak{g}}(e)$ where $\mathfrak{u} = \operatorname{im}(e) \cap \underbrace{\ker(e)}_{= Z_{\mathfrak{g}}(e)}$.

Proof Let (e,f,h) and (e,f',h') be two $\mathfrak{sl}_2$-triples.

1) If $h = h'$, then we claim that $f = f'$ as well. Note:
$$[f,e] = h = h' = [f',e] \Rightarrow [f-f',e] = 0 \Rightarrow f-f' \in \ker(\operatorname{ad} e) = Z_{\mathfrak{g}}(e).$$
But $f-f'$ has h-deg -2, so $f-f' \in Z_{\mathfrak{g}}(e)_{-2} = 0 \Rightarrow f = f'$.

2) Now $[h,e] = 2e = [h',e] \Rightarrow [h-h',e] = 0 \Rightarrow h-h' \in Z_{\mathfrak{g}}(e) = \ker(e)$.
But $h-h' = [f-f',e] \in \operatorname{im}(e)$. So $h-h' \in \operatorname{im}(e) \cap \ker(e) = \mathfrak{u}$.

3) The lemma implies that $\exists\, u \in U$ with $uhu^{-1} = h'$.
One easily checks that $(e, ufu^{-1}, uhu^{-1} = h')$ is an $\mathfrak{sl}_2$-triple,
so by part 1), $ufu^{-1} = f'$. $\square$

Def

Let (e,f,h) be an $\mathfrak{sl}_2$-triple. Write $G_{\mathfrak{sl}_2} = Z_G(\mathfrak{sl}_2)$ to be the simultaneous centralizer of (e,f,h) in G.

Prop

1) $G_{\mathfrak{sl}_2}$ is a maximal reductive subgroup in $Z_G(e)$.

2) U is the unipotent radical of $Z_G(e)$.

Proof

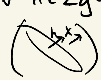
Omitted. $\square$

Sketch of proof of Jacobson-Morozov thm

1) First we want to shrink $Z_{\mathfrak{g}}(e)$ until it only contains nilpotent elements. We do this by repeatedly cutting out non-nilpotents. To $x \in Z_{\mathfrak{g}}(e)$, if $x \notin \mathcal{N}$, then Jordan form $x = s + n \Rightarrow [x,e]=0 \Rightarrow [s,e]=0 \Rightarrow s \in Z_{\mathfrak{g}}(e)$.
Then replace $\mathfrak{g}$ with $\underbrace{Z_{\mathfrak{g}}(s) \ni e}_{\text{reductive + proper}}$, or rather the semisimple part of $Z_{\mathfrak{g}}(s)$.
This shrinks the dimension, and removes x, so we're done by induction on dim.

2) We now find a semisimple h with $[h,e] = 2e$.
First take any $x \in \mathfrak{g}$ with $[x,e] = 2e$. Then $x = n + s$ (Jordan form).
But ad n is nilpotent, and e is an eigenvector for both ad n, ad s.
so $\operatorname{ad} n(e) = 0 \Rightarrow \operatorname{ad} x(e) = \operatorname{ad} s(e) = 2e$. So $h = s$ suffices.

3) Now we find f. Use h to get a grading. We want $f \in \mathfrak{g}_{-2}$ and $\operatorname{ad}(e)(f) = h$, i.e. we just need $h \in \operatorname{im}(e)$, then any $f \in (\operatorname{ad} e)^{-1}(h)$ suffices.
Arguing as in 3.2.16, $h \in \operatorname{im}(e) \longleftrightarrow \underset{\text{Killing form}}{k}(h, Z_{\mathfrak{g}}(e)) = 0 \longleftrightarrow \operatorname{Tr}(\operatorname{ad} h \circ \operatorname{ad} x) = 0 \quad \forall\, x \in Z_{\mathfrak{g}}(e)$.

But this is true b/c $Z_{\mathfrak{g}}(e)$ has only nilpotents in $\mathfrak{n}_+$, so . (i.e. ad h is $+ (\geq 0)$, ad x is $+ (>0)$, so nothing remaining to have a trace.)

$\square$