3.5 Goal: construct all irreps of $H(Z) \simeq \mathbb{Q}[W]$ in a purely geometric way.

Setup: $\begin{array}{c}\tilde{\mathcal{N}} \\ \mu \downarrow \\ \mathcal{N}\end{array}$, $Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$, for $Y \subset \mathcal{N}$, and

$$\begin{tikzcd} \tilde{Y} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \tilde{\mathcal{N}} \arrow[d] \\ Y \arrow[r, hook] & \mathcal{N} \end{tikzcd} \qquad Z_Y = \tilde{Y} \times_Y \tilde{Y}$$

Borel-Moore always with $\mathbb{Q}$-coefs

ex
$Z_x = \mathcal{B}_x \times \mathcal{B}_x$
for $x \in \mathcal{N}$

$H^{BM}(\mathcal{B}_x) =: H(\mathcal{B}_x)$

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$Z \circ Z_x = Z_x = Z_x \circ Z.$

Therefore, convolution gives us:

Lemma
$H(Z_x) := H^{BM}_{2 \cdot \dim \mathcal{B}_x}(Z_x)$ is an $H(Z)$-bimodule.

But Kunneth formula tells us that

$$H(Z_x) \simeq \underset{\substack{\circlearrowright \\ H(Z_x)}}{H(\mathcal{B}_x)} \otimes \underset{\substack{\circlearrowleft \\ H(Z_x)}}{H(\mathcal{B}_x)}, \qquad H(\mathcal{B}_x) := H^{BM}_{\dim \mathcal{B}_x}(\mathcal{B}_x)$$

From $Z \circ \mathcal{B}_x = \mathcal{B}_x = \mathcal{B}_x \circ Z$.

Prop
$H(\mathcal{B}_x)$ has a left-$H(Z)$-module and right-$H(Z)$-module structure, denoted by $H(\mathcal{B}_x)_L$, $H(\mathcal{B}_x)_R$.

Kunneth now tells us:

Prep

$$H(Z_x) \simeq H(B_x)_L \otimes H(B_x)_R \quad \text{as } H(Z)\text{-bimodules.}$$

Now, $G \curvearrowright N$ by conjugation, hence

$$g : x \mapsto gxg^{-1} \quad \Rightarrow \quad g : B_x \longrightarrow B_{gxg^{-1}},$$
$$b \mapsto gbg^{-1}.$$

Prep G-action and $H(Z)$-action on $H(B_x)$ commute:
$\forall\, g \in G, \quad z \in H(Z),$

$$\begin{tikzcd}
H(B_x) \arrow[r, "g"] \arrow[d, "z"'] & H(B_{gxg^{-1}}) \arrow[d, "z"] \\
H(B_x) \arrow[r, "g"] & H(B_{gxg^{-1}})
\end{tikzcd}.$$

Proof

$G \curvearrowright Z$, hence $G \longrightarrow Aut(H(Z))$.

So $\quad g(z) \cdot (\underset{H(B_{gxg^{-1}})}{\underset{\uparrow}{g(c)}}) = g(z \cdot c) = \downarrow\!\rightarrow$.

But G is connected, so G acts trivially on $H(Z)$,
hence $g(z) = z$ and the LHS is $z \cdot g(c) = \rightarrow\!\downarrow$.

$\square$

Now since $G_x = \left\{ \begin{matrix} \text{centralizer of} \\ x \in \mathcal{N} \end{matrix} \right\}$ preserves B_x,

$$G_x : B_x \longrightarrow B_x .$$

So the Prop tells us that G_x action and $H(Z)$ action commute on $H(B_x)$.

But $\underbrace{G_x^\circ}_{\text{conn. comp. of identity}}$ acts trivially on homology, so $C(x) := G_x / G_x^\circ = \pi_0(G_x)$ is the one we want:

Lemma

$C(x)$, $H(Z)$-action on $H(B_x)$ commute.
either left or right.

Since $C(x)$ is a finite group, and $H(B_x)$ is f.d. vector space,
semisimple

Prop Using $\mathbb{C}$-coefs,

$$H(B_x)_L \simeq \bigoplus_{\pi\text{-irrep of } C(x)} \pi \otimes H(B_x)_{L,\pi} ,$$

multiplicity space

and $H(B_x)_{L,\pi}$ are $H(Z)$-modules.

Remark

We should use $\mathbb{C}$ coefs to ensure semisimplicity, but actually $C(x)$-irreps are all defined over $\mathbb{Q}$ for most G anyway (everything except E_8).

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Def Let A be a $\mathbb{Q}$-alg, let M be a left A-mod.
Then $M^\vee := \mathrm{Hom}_{\mathbb{Q}}(M, \mathbb{Q})$ is a right A-mod.
$(\varphi \cdot a)(m) = \varphi(a \cdot m)$.

Claim $(H(B_x)_L)^\vee \simeq H(B_x)_R$ as right $H(Z)$-mod, with compatible $C(x)$-action.
$C(x) \curvearrowright (H(B_x)_L)^\vee$ by $(g \cdot \varphi)(h) = \varphi(g^{-1} \cdot h)$

Proof deferred to §3.6. $\square$

MAIN THM

1) $H(B_x)_\pi$ is a simple $H(Z)_{\mathbb{C}}$-mod $\forall\, x \in \mathcal{N},\ \pi \in \mathrm{Irrep}(C(x))$
2) $H(B_x)_\pi \underset{H(Z)\text{-mod}}{\cong} H(B_y)_\psi \longleftrightarrow (x, \pi) \underset{G\text{-conj.}}{\sim} (y, \psi)$.
3) All simple (complex) $H(Z)_{\mathbb{C}}$-mod arise in this way.

In other words,
$$\Big\{ H(B_x, \mathbb{C})_\pi \ \Big|\ G\text{-conj. classes of pairs } (\underset{\in \mathcal{N}}{x}, \underset{\text{irrep of } C(x)}{\pi}) \Big\}$$
forms a complete set of irreps of $H(Z, \mathbb{C}) \simeq \mathbb{C}[W]$.

Ex If $x \in \mathcal{N}^{reg}$, then $B_x = \{pt\}$ b/c $\exists$ unique borel $\mathfrak{b} \ni x$.
Then $H(B_x) = H_0^{BM}(pt) = \mathbb{Q}$, which corresponds to the trivial W-rep.