1) Steinberg variety

<u>Def</u> The <u>Steinberg</u> variety is

$$Z := \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} = \{(x, \mathfrak{b}, \mathfrak{b}') \mid x \in \mathfrak{b} \cap \mathfrak{b}'\}$$

<u>Prop</u> $Z = \bigsqcup_{Y_w - \text{ G-orbits in } B\times B} T^*_{Y_w}(B\times B)$ is the union of conormal bundles to G-orbits in $B \times B$.

<u>Proof</u> Let $Y_w = G\cdot(\mathfrak{b}_1, \mathfrak{b}_2)$. Then $T_{Y_w}(B\times B)_{(\mathfrak{b}_1,\mathfrak{b}_2)} = \{(u\cdot\mathfrak{b}_1, u\cdot\mathfrak{b}_2) \mid u \in \mathfrak{g}\}$.

Then $T^*_{Y_w}(B\times B) = \{\alpha = (x_1, \mathfrak{b}_1, x_2, \mathfrak{b}_2)\}$ annihilating $T_{Y_w}(B\times B)_{(\mathfrak{b}_1,\mathfrak{b}_2)}$.
$x_1, x_2 \in \mathfrak{g}^*,\ x_i \in \mathfrak{b}_i^\perp$

This comes out to $\langle x_1, u\rangle + \langle x_2, u\rangle = 0 \Rightarrow x_1 = -x_2$.

This is just Z. $\square$

<u>Cor</u> $Z = \bigsqcup_{w\in W} T^*_{Y_w}(B\times B)$.

Irred. comp. of Z are $\overline{T^*_{Y_w}(B\times B)}$, indexed by W.

2) Irred. comps of Steinberg

For any adjoint orbit $\mathbb{O} \subset \mathcal{N}$, take $Z_0 \subset Z$ to be lying over $\mathbb{O}$:

$$\begin{tikzcd} Z_0 \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Z \arrow[d] \\ \mathbb{O} \arrow[r, hook] & \mathcal{N} \end{tikzcd}.$$

Prop Z_0 is pure dim; $\dim Z_0 = \dim Z = 2 \cdot \dim \mathfrak{n} = 2 \cdot \dim B$.

Proof

1) Let $\tilde{\mathbb{O}} = \mu^{-1}(\mathbb{O})$ in $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$.

Then $\tilde{\mathbb{O}} \simeq G \times_B (\mathbb{O} \cap \mathfrak{n})$ is a fibration over G/B, so pure dim.

pure dim $= \frac{1}{2} \dim \mathbb{O}$

2) $Z_0 = \tilde{\mathbb{O}} \times_{\mathbb{O}} \tilde{\mathbb{O}}$, so Z_0 is pure dim, and dim is

$$\dim Z_0 = 2 \dim \tilde{\mathbb{O}} - \dim \mathbb{O} = 2 \dim B + \dim \mathbb{O} - \dim \mathbb{O} = 2 \dim B = 2 \dim \mathfrak{n}.$$

independent of $\mathbb{O}$. $\square$

Cor Irred. comp. of Z:

1) $\bigsqcup_{w \in W} \overline{T^*_{Y_w}(B \times B)}$.

2) $\bigsqcup (\mathbb{O}, \text{(irred comp. of } Z_0))$

nilpotent orbit in $\mathcal{N}$

Now we need to study this!

$$\begin{tikzcd} \tilde{\mathcal{N}} \arrow[d] & \supset & \tilde{\mathbb{O}} \arrow[d] & & \dim \mathfrak{n} + \frac{1}{2} \dim \mathbb{O} \\ \mathcal{N} & \supset & \mathbb{O} & & \dim \mathbb{O} = 2 \dim \mathfrak{n} - \dim B_x \\ Z & \supset & Z_0 \arrow[u] & & 2 \dim \mathfrak{n} \end{tikzcd}$$

B_x = Springer fiber over $x \in \mathcal{N}$

3) Irred Components of Z_0

Fix $x \in \mathbb{O} \subset \mathcal{N}$.

$$\mu^{-1}(\mathbb{O}) = \tilde{\mathbb{O}} = G \times_B (\mathbb{O} \cap \mathfrak{n}) = G \times_{G_x} B_x \qquad \begin{array}{ccc} \tilde{\mathbb{O}} & = & G \times_{G_x} B_x \\ \downarrow & & \downarrow \\ \mathbb{O} & = & G/G_x \end{array}$$

map

projection.

springer fiber

$$\Rightarrow \quad Z_0 = \tilde{\mathbb{O}} \times_{\mathbb{O}} \tilde{\mathbb{O}} = G \times_{G_x} (B_x \times B_x) \qquad \text{for } x \in \mathbb{O}$$

stabilizer

<u>Cor</u> Irred. comp. of Z_0 are $G \times_{G_x} (B_1 \times B_2)$ where B_1, B_2 are irred. comp. of B_x.

<u>Cor</u> B_x is <u>pure</u> dim; $\dim B_x = \underset{\dim \mathfrak{n}}{\underset{\shortparallel}{\dim B}} - \frac{1}{2} \dim \mathbb{O}$.

<u>Remark</u> B_x is also connected by Zariski main thm.

<u>Def</u> Let G_x° = conn. comp. of identity in G_x. Let $C(x) = G_x / G_x^\circ$.

Then $C(x) \curvearrowright \{\text{irred. comp. of } B_x\} = \{B_x^\alpha\}$.

<u>Prop</u> The irred. comp. of Z_0 are $Z_0^{\alpha,\beta} = G \times_{G_x} (B_x^\alpha \times B_x^\beta)$.

$$\{Z_0^{\alpha,\beta}\} \longleftrightarrow \{C(x)\text{-orbits on pairs of irred. comps of } B_x\}.$$

<u>Cor</u> $\mathcal{N}$ has finitely many G-orbits, i.e. conjugacy classes.

<u>Proof</u> $Z = \bigcup_{\mathbb{O}} Z_0$, but Z_0 is pure-dim of $\dim Z$, so finitely many $\mathbb{O}$. $\square$