1) Springer resolution

Def $\mathcal{N} \subset \mathfrak{g}$ is the nilpotent cone, i.e., set of all nilpotents in $\mathfrak{g}$.
This is singular only at 0.

Def $\tilde{\mathcal{N}}$ is defined by

$$\begin{tikzcd} \mu^{-1}(\mathcal{N}) \arrow[r, equal] & \tilde{\mathcal{N}} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \tilde{\mathfrak{g}} \arrow[d, "\mu"] \\ & \mathcal{N} \arrow[r, hook] & \mathfrak{g} \end{tikzcd}.$$

$\tilde{\mathcal{N}} = \{ (x, \mathfrak{b}) \mid x \in \mathfrak{b} \text{ and } x \in \mathcal{N} \}$

Prop $\tilde{\mathcal{N}} \simeq G \times_B \mathfrak{n}$
as G-equivariant vector bundles.

Proof basically b/c $x \in \mathfrak{b} \cap \mathcal{N} \Longleftrightarrow x \in \mathfrak{n}$, and all things respect G-conjugation.

Cor $\tilde{\mathcal{N}}$ smooth

Now identify $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ by Killing form.

Because $\mathfrak{g}^* \supset \mathfrak{b}^\perp \longleftrightarrow \mathfrak{n} \subset \mathfrak{g}$, we get

Prop $$T^*\mathcal{B} = G \times_B \mathfrak{b}^\perp = G \times_B \mathfrak{n} = \tilde{\mathcal{N}}.$$

Since the moment map on $T^*\mathcal{B}$ is $(g, x) \longmapsto x$,
the moment map for $T^*\mathcal{B}$ matches $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$.

Def Therefore, map $\mu : T^*\mathcal{B} = \tilde{\mathcal{N}} \to \mathcal{N}$ is called Springer resolution.

2) Intrinsically defining nilpotence

$\mathfrak{g} \longleftrightarrow \mathfrak{g}^*$ via Killing form.

Def $x^* \in \mathfrak{g}^*$ is nilpotent if $x \in \mathfrak{g}$ is nilpotent.

This is not intrinsic; depends on iso $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$.

Def Let $\mathbb{C}[\mathfrak{g}]^G_+ \subset \mathbb{C}[\mathfrak{g}]^G$ be the ideal of poly's with no constant term.

Prop (Kostant)
$x \in \mathfrak{g}$ is nilpotent $\longleftrightarrow$ $P(x)=0 \ \forall \ P \in \mathbb{C}[\mathfrak{g}]^G_+$.

Cor $x \in \mathfrak{g}^*$ is nilpotent defined intrinsically.

Cor We have diagram

$$\begin{tikzcd}
 & \tilde{\mathfrak{g}} \ (x,\mathfrak{b}) \arrow[dl] \arrow[dr] & \\
x \ \mathfrak{g} \arrow[dr] & & \mathfrak{h} \ x \bmod [\mathfrak{b},\mathfrak{b}] \arrow[dl] \\
 & \mathfrak{g}/G \simeq \mathfrak{h}/W \ [x] &
\end{tikzcd}$$

This is not Cartesian, but it is when restricted to the regular locus.

Then we can upgrade to:

$$\begin{tikzcd}
\tilde{\mathcal{N}} = \mu^{-1}(\mathcal{N}) \arrow[r, hook] \arrow[d, "\mu"] & \tilde{\mathfrak{g}} \arrow[d, "\mu"] \arrow[dr, "\nu"] & \\
\rho^{-1}(0) = \mathcal{N} \arrow[r, hook] \arrow[d] & \mathfrak{g} \arrow[dr, "\rho"] & \mathfrak{h} \arrow[d, "\pi"] \\
\{0\} \arrow[rr, hook] & & \mathfrak{h}/W
\end{tikzcd}$$

$\rho^{-1}(0)$ $=$ nilpotents

Cor (Kostant)
$\mathcal{N}$ irred with $\dim \mathcal{N} = 2 \cdot \dim \mathfrak{n}$.

Proof

$$\begin{tikzcd}
T^*\mathcal{B} = \tilde{\mathcal{N}} - 2\cdot\dim\mathfrak{n} \arrow[d] \\
\mathcal{N}
\end{tikzcd}$$

$\Rightarrow 2\cdot\dim\mathfrak{n} \geq \dim \mathcal{N}$.

Also,

$$\begin{tikzcd}
\mathcal{N} \arrow[r] \arrow[d] & \mathfrak{g} \arrow[d] \\
0 \arrow[r] & \mathfrak{h}/W
\end{tikzcd}$$

so $\dim \mathcal{N} - \dim\{0\} \geq \dim \mathfrak{g} - \dim \mathfrak{h}/W$
$= 2\cdot\dim\mathfrak{n}$. $\square$

3) Nilpotent orbits $G \curvearrowright \mathcal{N}$

Prop The number of nilpotent conjugacy classes is finite.

Proof postponed to next chapter.

Prop Regular nilpotent elements form unique dense open orbit.

Proof
- finitely many orbits $\Rightarrow$ unique dense open orbit.
- $x \in \mathcal{N}^{reg} \Rightarrow \dim G_x \overset{\text{-stab}}{=} \dim Z_{\mathfrak{g}}(x) = \dim \mathfrak{h} \Rightarrow \dim G.x = \dim G - \dim G_x = 2 \dim \mathfrak{n}$
$= \dim \mathcal{N}$. □

Now fix $T, U \subset B \subset G$.

We have $\mathfrak{n} = \bigoplus \mathbb{C} \cdot e_\alpha$ α positive roots.

Let Δ be simple roots.

Then $\mathfrak{n}^{reg} = \{ \sum \lambda_\alpha e_\alpha \mid \alpha \in \Phi^+, \lambda_\alpha \neq 0 \text{ for } \alpha \in \Delta \}$

$\simeq (\mathbb{C}^\times)^{|\Delta|} \times (\mathbb{C})^{|\Phi^+ \setminus \Delta|}$.

In fact, $\mathfrak{n}^{reg}$ consists only of regular nilpotents

Cor $e_{\alpha_1} + \cdots + e_{\alpha_n}$ for $\alpha_i \in \Delta$ is nilpotent regular.

Prop Any regular nilpotent is contained in a unique Borel.

Cor $\widetilde{\mathcal{N}} \to \mathcal{N}$ is an iso on the open orbit,

hence is a resolution of singularities.

Thm $\mathcal{N} = \bigsqcup_{G\text{-orbits}} \mathbb{O}$ is an algebraic stratification.