

1) Springer resolution

Def  $\mathcal{N} \subset \mathfrak{g}$  is the nilpotent cone, i.e., set of all nilpotents in  $\mathfrak{g}$ .  
This is singular only at  $0$ .

Def  $\tilde{\mathcal{N}}$  is defined by

$$\begin{array}{ccc} \tilde{\mathcal{N}} & \hookrightarrow & \tilde{\mathfrak{g}} \\ \mu^{-1}(x) \cong & \downarrow & \downarrow \mu \\ \mathcal{N} & \hookrightarrow & \mathfrak{g} \end{array} \quad \tilde{\mathcal{N}} = \{ (x, b) \mid x \in \mathcal{N} \text{ and } x \in \mathfrak{b} \}$$

Prop  $\tilde{\mathcal{N}} \cong G \times_B \mathfrak{n}$   
as  $G$ -equivariant vector bundles.

Proof basically b/c  $x \in \mathfrak{b} \cap \mathcal{N} \iff x \in \mathfrak{n}$ , and all things respect  $G$ -conjugation.

Cor  $\tilde{\mathcal{N}}$  smooth

Now identify  $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$  by Killing form.

Because  $\mathfrak{g}^* = \mathfrak{b}^\perp \iff \mathfrak{n} \subset \mathfrak{g}$ , we get

Prop  $T^*B = G \times_B \mathfrak{b}^\perp = G \times_B \mathfrak{n} = \tilde{\mathcal{N}}$ .

Since the moment map on  $T^*B$  is  $(g, x) \mapsto x$ ,  
the moment map for  $T^*B$  matches  $\mu: \tilde{\mathcal{N}} \rightarrow \mathcal{N}$ .

Def Therefore, map  $\mu: T^*B = \tilde{\mathcal{N}} \rightarrow \mathcal{N}$  is called Springer resolution.

2) Intrinsically defining nilpotence

$$\mathfrak{g} \longleftrightarrow \mathfrak{g}^* \text{ via Killing form.}$$

Def  $x \in \mathfrak{g}^*$  is nilpotent if  $x \in \mathfrak{g}$  is nilpotent.

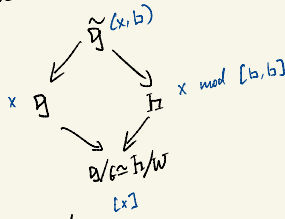
This is not intrinsic; depends on iso  $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ .

Def Let  $\mathbb{C}[\mathfrak{g}]_+^G \subset \mathbb{C}[\mathfrak{g}]^G$  be the ideal of poly's with no constant term.

Prop (Kostant)  $x \in \mathfrak{g}$  is nilpotent  $\iff P(x) = 0 \forall P \in \mathbb{C}[\mathfrak{g}]_+^G$ .

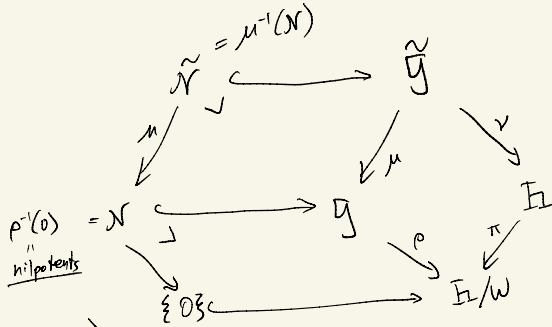
Cor  $x \in \mathfrak{g}^*$  is nilpotent defined intrinsically.

Cor We have diagram



This is not Cartesian, but it is when restricted to the regular locus.

Then we can upgrade to:



Cor (Kostant)

$\mathfrak{N}$  irred with  $\dim \mathfrak{N} = 2 \cdot \dim \mathfrak{N}$ .

Proof

$$\begin{array}{c}
 T^*B = \tilde{\mathfrak{N}} - 2 \cdot \dim \mathfrak{N} \\
 \downarrow \\
 \mathfrak{N}
 \end{array}
 \Rightarrow 2 \cdot \dim \mathfrak{N} \geq \dim \mathfrak{N}$$

Also,

$$\begin{array}{ccc}
 \mathfrak{N} & \longrightarrow & \mathfrak{g} \\
 \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathbb{R}/w
 \end{array}$$

$$\text{so } \dim \mathfrak{N} - \dim \{0\} \geq \dim \mathfrak{g} - \dim \mathbb{R}/w = 2 \cdot \dim \mathfrak{N}$$

□

### 3) Nilpotent orbits $G \curvearrowright \mathcal{N}$

Prop The number of nilpotent conjugacy classes is finite.

Proof postponed to next chapter.

Prop Regular nilpotent elements form unique dense open orbit.

Proof - finitely many orbits  $\Rightarrow$  unique dense open orbit.  
 -  $x \in \mathcal{X}^{\text{reg}} \Rightarrow \dim G_x^{\text{-stab}} = \dim Z_{\mathfrak{g}}(x) = \dim \mathfrak{h} \Rightarrow \dim G \cdot x = \dim G - \dim G_x = 2 \dim \mathfrak{n} = \dim \mathcal{N}. \square$

Now fix  $T \cup B \subset G$ .

We have  $\mathfrak{n} = \bigoplus \mathbb{C} \cdot e_{\alpha}$   $\alpha$  positive roots.

Let  $\Delta$  be simple roots.

Then  $\mathfrak{n}^{\text{reg}} = \left\{ \sum \lambda_{\alpha} e_{\alpha} \mid \alpha \in \Phi^+, \lambda_{\alpha} \neq 0 \text{ for } \alpha \in \Delta \right\}$   
 $\cong (\mathbb{C}^{\times})^{|\Delta|} \times (\mathbb{C})^{|\Phi^+ - \Delta|}$

In fact,  $\mathfrak{n}^{\text{reg}}$  consists only of regular nilpotents

Cor  $e_{\alpha_1} + \dots + e_{\alpha_n}$  for  $\alpha_i \in \Delta$  is nilpotent regular.

Prop Any regular nilpotent is contained in a unique Borel.

Cor  $\tilde{\mathcal{N}} \rightarrow \mathcal{N}$  is an iso on the open orbit,  
 hence is a resolution of singularities.

Thm  $\mathcal{N} = \bigsqcup_{G\text{-orbits}} \mathcal{O}$  is an algebraic stratification.