2.1 Hilbert's Nullstellensatz

Let A be an associative $\mathbb{C}$-algebra with unit, not necessarily commutative.

Definition 1.

For $a \in A$, define

$$\operatorname{Spec} a := \{\lambda \in \mathbb{C} \mid a - \lambda \text{ is not invertible}\}.$$

Remark 2.

The motivation comes from many places. If $A = \mathrm{Mat}_n(\mathbb{C})$, then $\operatorname{Spec} a$ is just the set of eigenvalues of a, equivalent to roots of its characteristic polynomial. If A is the algebra of continuous functions $[0,1] \to \mathbb{C}$, then $\operatorname{Spec} a$ is just the image of a.

Theorem 3 (Nullstellensatz)**.**

Assume that $\dim_{\mathbb{C}} A$ is countable. Then for all $a \in A$ we have $\operatorname{Spec} a \neq \emptyset$. Also, a is nilpotent iff $\operatorname{Spec} a = \{0\}$.

Proof.

Suppose for the sake of contradiction that $\operatorname{Spec} a = \emptyset$. Then $(a - \lambda)$ is always invertible and $\{(a-\lambda)^{-1} \mid \lambda \in \mathbb{C}\}$ forms an uncountable family of elements of A, hence there must be some finite nonempty subset which is linearly dependent. But then we just clear denominators and find some polynomial $P(t) \in \mathbb{C}[t]$ such that $P(a) = 0$. Thus we can factor P and get

$$(a - \lambda_1)(a - \lambda_2) \cdots (a - \lambda_n) = 0,$$

contradicting invertibility of $(a - \lambda)$ for all λ.

Now if a is nilpotent, then $a^n = 0$ so $0 \in \operatorname{Spec} a$. For $\lambda \notin \mathbb{C}$ we can write $(a - \lambda)^{-1}$ as an infinite geometric series in a - but since a is nilpotent, this series terminates and the inverse is well-defined as a polynomial. It follows that $a - \lambda$ is invertible.

If $\operatorname{Spec} a = \{0\}$, then we just repeat the first argument (since $\mathbb{C} \setminus \{0\}$ is still uncountable). Then we find that

$$a^n (a - \lambda_1) \cdots (a - \lambda_m) = 0.$$

Since we assumed that $a - \lambda_i$ are invertible, we find that $a^n = 0$.

□

Corollary 4.

If A is a skew field of at most countable dimension over $\mathbb{C}$, then $A = \mathbb{C}$.

Proof.
We have $\mathbb{C} \hookrightarrow A$; pick $a \in A \setminus \mathbb{C}$. Then for any $\lambda \in \mathbb{C}$, we have $a - \lambda \neq 0$, hence is invertible. But then $\operatorname{Spec} a = \emptyset$, contradicting Theorem 3 (Nullstellensatz).

□

Lemma 5 (Schur)**.**

Let M be a simple A-module. Then $\operatorname{End}_A(M)$ is a skew field.

Proof.
Let $0 \neq f \in \operatorname{End}_A(M)$. Then $\ker f$ and $\operatorname{im} f$ are A-submodules of the simple module M, hence $\ker f = 0$ and $\operatorname{im} f = M$, so M is invertible.

□

Corollary 6.

Let M be a simple A-module. If $\dim_{\mathbb{C}} A$ is countable, then $\operatorname{End}_A(M) = \mathbb{C}$.

Proof.
First, M is simple, so $A \cdot m = M$ or 0 for any $0 \neq m \in M$. Picking m such that $A \cdot m = M$, we have $A \twoheadrightarrow M$, hence $\dim_{\mathbb{C}} M$ is also countable.

Next, for m such that $A \cdot m = M$, it follows that any endomorphism $f \in \operatorname{End}_A(M)$ is uniquely determined by $f(m) \in M$. Thus $\dim_{\mathbb{C}} \operatorname{End}_A(M) \leq \dim_{\mathbb{C}} M$ is countable.

Now Lemma 5 (Schur) implies that $\operatorname{End}_A(M)$ is a skew field, and Corollary 4 implies that it's actually $\mathbb{C}$.

□

Corollary 7.

Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra. Let $\mathcal{U}(\mathfrak{g})$ be its universal enveloping algebra.
Let $Z(\mathfrak{g})$ be the center of $\mathcal{U}(\mathfrak{g})$.
For any simple $\mathfrak{g}$-module M, the center $Z(\mathfrak{g})$ acts by scalars.

Proof.
$\mathcal{U}(\mathfrak{g})$ has a PBW basis, which is countable, hence has countable dimension. Then Corollary 6 implies that $\mathrm{End}_{\mathcal{U}(\mathfrak{g})}(M) = \mathbb{C}$.

By definition $Z(\mathfrak{g})$ commutes with elements of $\mathcal{U}(\mathfrak{g})$, so we have a map

$$Z(\mathfrak{g}) \to \mathrm{End}_{\mathcal{U}(\mathfrak{g})}(M) = \mathbb{C}.$$

□

2.2 Affine Algebraic Varieties

We'll actually skip most of this section until Poincaré series, since it's basically just review of algebraic geometry.

Let $E = \bigoplus_{i \geq 0} E_i$ be a graded vector space with each E_i being finite-dimensional.

Definition 8 (graded dual)**.**

The **graded dual** of E is defined to be

$$E^* := \bigoplus_{i \geq 0} (E_i)^*.$$

For a graded $F \subset E$, we have

$$F^\perp := \mathrm{Ann}_{E^*}(F) \simeq (E/F)^*.$$

We also have the notion of the tensor product of graded vector spaces, given by

$$(E \otimes E')_k = \bigoplus_{i+j=k} E_i \otimes E'_j.$$

Remark 9.

The hypothesis that the graded components are bounded below is crucial so that the graded components of tensor products are still finite-dimensional.

Definition 10 (Poincaré series)**.**

To a graded vector space E, we define its **Poincaré series** $P(E)$ as the formal power series

$$P(E) = \sum_{i=0}^{\infty} (\dim E_i) t^i.$$

Although the vector spaces we are working with are now infinite-dimensional, the key condition that the graded components are all finite-dimensional allows us to essentially work with dimension arguments as usual.

Lemma 11.

1. $P(E \otimes E') = P(E) \cdot P(E')$.
2. $P(E/F) = P(E) - P(F)$.
3. $P(E^*) = P(E)$.

2.3 The Deformation Construction

Lattices

Let A be a ring with unit and $t \in Z(A)$ a central element which is not a zero-divisor, such that

$$\bigcap_{i \geq 1} t^i A = \{0\}.$$

Let $A_t = A[t^{-1}]$ be the localization of A at t. Let M be a finitely generated A_t-module.

Definition 12 (lattice)**.**

A **lattice in** M is a finitely generated A-submodule $L \subset M$ such that $A_t \cdot L = M$.

An equivalent characterization is that $\bigcup_{k \geq 0} t^{-k} L = M$, so that the missing elements get "filled in" by the t^{-1}-action on L. (Perhaps if t is associated with "somewhat large," then $t^{-k}L$ essentially fills in the gaps, so L indeed looks like a lattice in M.)

Example 13.

Let $A = \mathbb{C}[t]$ and $M = A_t = \mathbb{C}[t, t^{-1}]$. Then $L = t^n \mathbb{C}[t]$ is a lattice for any $n \in \mathbb{Z}$.

Proposition 14.

For any two lattices L, L' there are nonnegative integers $a, b \geq 0$ such that

$$t^a \cdot L \subset L' \subset t^{-b} \cdot L.$$

Proof.
Since L' is finitely generated, pick a finite set of generators $u_1, \dots, u_r$. Then each $u_i \in M = \bigcup_{k \geq 0} t^{-k} \cdot L$, so there exists $b \geq 0$ such that $u_i \in t^{-b} \cdot L$ for all i. It follows that $L' \subset t^{-b} \cdot L$. The other direction follows by symmetry.

□

From now on, **assume that A is noetherian**.

Lemma 15.

Suppose we have a short exact sequence of A_t-modules $0 \to M' \to M \to M'' \to 0$. Let $L \subset M$ be a lattice. Then:

1. We get lattices $L' := L \cap M'$ in M' and $L'' := L/(L \cap M')$ in M''.
2. $0 \to L' \to L \to L'' \to 0$ is exact.

Let $K^+(A)$ denote the Grothendieck semigroup of finitely-generated A-modules; let $K(A)$ be the Grothendieck group.

Lemma 16.

For any two lattices $L, L' \subset M$, we have that

$$[L/t \cdot L] = [L'/t \cdot L'] \in K^+(A/t \cdot A).$$

Proof.
The basic idea is to check the statement when L, L' are "adjacent" lattices, then extend one step at a time.

Adjacent means that

$$t \cdot L' \subset t \cdot L \subset L' \subset L.$$

Then we form the short exact sequences

$$0 \to L'/t \cdot L \to L/t \cdot L \to L/L' \to 0,$$
$$0 \to t \cdot L/t \cdot L' \to L'/t \cdot L' \to L'/t \cdot L \to 0$$

which gives us

$$[L/t \cdot L] = [L'/t \cdot L] + [L'/t \cdot L],$$
$$[L'/t \cdot L'] = [t \cdot L/t \cdot L'] + [L'/t \cdot L].$$

Finally, it suffices to note that $t \cdot L/t \cdot L' \cong L/L'$. Thus if L, L' are adjacent then we have the result. Now we just need to construct adjacent lattices; we can do this by taking $L_j := L + t^j \cdot L'$

; then L_j, L_{j+1} are adjacent, and for large j, $L_j = L$, while for small j, $L_j = t^j \cdot L'$.

□

Rees algebra

Let B be a ring with a separating $\mathbb{Z}$-filtration:

$$\cdots \subset B_{-1} \subset B_0 \subset B_1 \subset \cdots,$$

with $\bigcup_{n\in\mathbb{Z}} B_n = B$, $\bigcap_{n\in\mathbb{Z}} B_n = 0$, and $1 \in B_0$; we also have $B_i \cdot B_j \subset B_{i+j}$.

Definition 17 (Rees algebra)**.**

Define the **Rees algebra** of B with respect to the filtration above to be

$$\widehat{B} := \sum_{n\in\mathbb{Z}} B_n \cdot t^n \subset B[t, t^{-1}].$$

The following statements are fairly straightforward.

Proposition 18.

1. $\widehat{B}$ is a subring of $B[t, t^{-1}]$.
2. $\widehat{B}$ is a $\mathbb{Z}$-graded ring, graded by the powers of t, so that $(\widehat{B})_i \cong B_i$.
3. $t \in \widehat{B}$ is central and not a zero-divisor.
4. $\bigcup_{k\geq 0} t^{-k}\widehat{B} = B[t, t^{-1}]$.
5. $\widehat{B}/t \cdot \widehat{B} \cong \operatorname{gr} B$.
6. $(\widehat{B})_t = B[t, t^{-1}]$.

This has a natural geometric interpretation. Let $X = \operatorname{Spec} B$ where B has a $\mathbb{Z}$-filtration as above. Then consider $\widehat{X} := \operatorname{Spec} \widehat{B}$. We have a natural map $\mathbb{C}[t] \hookrightarrow \widehat{B}$, giving us a surjection

$$f : \widehat{X} \twoheadrightarrow \mathbb{A}^1.$$

Since t is not a zero-divisor, $\widehat{B}$ is flat over $\mathbb{C}[t]$, so we have a flat family $\widehat{X}$ over $\mathbb{A}^1$. Let's identify $\mathbb{A}^1 \simeq \mathbb{C}$. Then the fiber over 0 is

$$f^{-1}(0) = \operatorname{Spec} \widehat{B}/t \cdot \widehat{B} = \operatorname{Spec} \operatorname{gr} B.$$

On the other hand, the fibers over $\mathbb{C}^\times$ are all just copies of X:

$$f^{-1}(\mathbb{C}^\times) = \operatorname{Spec} (\widehat{B})_t = \operatorname{Spec} B[t, t^{-1}] = X \times \mathbb{C}^\times.$$

We have an interesting picture when we choose a specific filtration for B. Let $Y \subset X$ be a smooth closed subvariety, and let $\mathcal{I}_Y \subset \mathcal{O}(X) = B$ be the defining ideal of Y. Let's consider the $\mathbb{Z}$-filtration

$$\dots, \underbrace{\mathcal{I}_Y^4}_{B_{-4}}, \underbrace{\mathcal{I}_Y^3}_{B_{-3}}, \underbrace{\mathcal{I}_Y^2}_{B_{-2}}, \underbrace{\mathcal{I}_Y}_{B_{-1}}, \underbrace{\mathcal{O}(X)}_{B_0}, \underbrace{\mathcal{O}(X)}_{B_1}, \underbrace{\mathcal{O}(X)}_{B_2} \dots$$

Then we have

$$\operatorname{gr} B = \bigoplus_{k \geq 0} \mathcal{I}_Y^k / \mathcal{I}_Y^{k+1} = \bigoplus_{k \geq 0} S^k(\mathcal{I}_Y / \mathcal{I}_Y^2) = \mathcal{O}(T_Y X),$$

where $T_Y X$ is the **normal bundle** of Y in X. On the other hand,

$$(\widehat{B})_t = B[t, t^{-1}] = B \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}].$$

So once again we have a flat family Spec $\widehat{B}$ over $\mathbb{A}^1 \simeq \mathbb{C}$; here, the special fiber is the normal bundle

$$f^{-1}(0) = T_Y X,$$

while the remaining fibers are all $X = \operatorname{Spec} B$:

$$f^{-1}(\mathbb{C}^\times) = \operatorname{Spec} (\widehat{B})_t = X \times \mathbb{C}^\times.$$

We can also put filtrations on the modules. Suppose M is a finitely generated B-module, with a separating $\mathbb{Z}$-filtration (i.e., $\bigcap M_i = 0$) compatible with the $\mathbb{Z}$-filtration on B (i.e., $B_i \cdot M_j \subset M_{i+j}$). Then $\operatorname{gr} M$ is a $\operatorname{gr} B$-module. We can construct the **Rees module** of M in the same way:

$$\widehat{M} := \sum_{n \in \mathbb{Z}} M_n \cdot t^n \subset M[t, t^{-1}].$$

Clearly $\widehat{M}$ is a graded $\widehat{B}$-module, and $\widehat{M}/t \cdot \widehat{M} = \operatorname{gr} M$.

Definition 19 (good filtration)**.**

A filtration $\{M_n\}$ on M is a **good filtration** if it satisfies one of the following equivalent conditions:

1. M is finitely generated in a strong sense: there are $m_1, \dots, m_r \in M$ so that $M_n = \sum_{i=1}^r B_{n+k_i} \cdot m_i$ for $k_1, \dots, k_r \in \mathbb{Z}$.
2. $\widehat{M}$ is $\widehat{B}$-lattice in $M[t, t^{-1}]$.
   If $B_i = 0$ for all $i < 0$, then there's another equivalent formulation.
3. $\operatorname{gr} M$ is finitely generated over $\operatorname{gr} B$.

Good filtrations are really easy to construct in practice; for example, we can define M_n via condition 1 of Definition 19 (good filtration) by picking some generators, and this automatically constructs a filtration on a finitely generated module M. The only downside is that this is far from canonical, which is why we need to pass to K-theory.

Corollary 20.

The class $[\operatorname{gr} M] \in K^+(\operatorname{gr} B)$ does not depend on the choice of good filtration.

Proof.
Use Lemma 16. We have $\operatorname{gr} B = \widehat{B}/t \cdot \widehat{B}$, and for a module M with filtrations M_i and M_i', we have $[\widehat{M}/t \cdot \widehat{M}] = [\widehat{M'}/t \cdot \widehat{M'}] = [\operatorname{gr} M]$.

□

Additionally, many properties can be checked on K-theory.

Proposition 21.

Suppose M, N are filtered B-modules such that $M_i = N_i = 0$ for $i \ll 0$. Then:

1. If $\operatorname{gr} M$ is a free $\operatorname{gr} B$-module of rank r, then M is a free B-module of rank r.
2. If $\phi : M \to N$ is a map of filtered B-modules such that the induced map on associated graded is an isomorphism, then ϕ itself was already an isomorphism.

Specialization in K-theory

As in the setup in Rees algebra, let B be a ring, flat over $\mathbb{C}[t]$, and let $X = \operatorname{Spec} B$. Then we have the map $f : X \to \mathbb{A}^1 \simeq \mathbb{C}$, with special fiber

$$X_0 := f^{-1}(0) = \operatorname{Spec} B/t \cdot B$$

and the preimage over the open set $\mathbb{C}^\times$

$$X^\times := f^{-1}(\mathbb{C}^\times) = \operatorname{Spec} B_t.$$

We want to define a **specialization morphism**

$$\lim_{t \to 0} : K^+(X^\times) \to K^+(X_0),$$

"specializing" a module on the dense open subset to the special fiber. We will define it as follows. Let $M \in K^+(X^\times)$ be a finitely generated B_t-module, and choose any lattice $L \subset M$. Then by definition L is a finitely generated B-module. Therefore $L/t \cdot L$ is a finitely generated

$B/t \cdot B$-module, and Lemma 16 implies that the class $[L/t \cdot L] \in K^+(X_0)$ is independent of the choice of lattice L. We declare that

$$\lim_{t \to 0}[M] = [L/t \cdot L],$$

and extend linearly.

Example 22.

Suppose $\mathcal{F}$ is a coherent sheaf on X, flat over $\mathbb{A}^1$. Then let $\mathcal{F}^\times := \mathcal{F}|_{X^\times}$ and $\mathcal{F}_0 := \mathcal{F}|_{X_0}$. We have that

$$\lim_{t \to 0} \mathcal{F}^\times = \mathcal{F}_0.$$

(This is just translating the algebraic statement that M is a lattice for itself, hence $\lim_{t \to 0}[M] = [M/t \cdot M] = [M \otimes_B B/t \cdot B]$.)

2.4 $\mathbb{C}^\times$-actions on a projective variety

Let X be a smooth complex projective variety with an $\mathbb{G}_m$-action (recall that $\mathbb{G}_m \simeq \mathbb{C}^\times$). Embed $\mathbb{G}_m \simeq \mathbb{C}^\times \hookrightarrow \mathbb{CP}^1$, so that the complement consists of exactly $\{0, \infty\}$.

Lemma 23.

For every $x \in X$, the map $z \mapsto z \cdot x$ has a limit as $\mathbb{C}^\times \ni z \to 0, \infty \in \mathbb{CP}^1$. Furthermore, the limit points are precisely the fixed points of the $\mathbb{C}^\times$-action.

Corollary 24.

The set of $\mathbb{C}^\times$-fixed points on X is always nonempty.

Let $\mathbb{W}$ denote the $\mathbb{C}^\times$-fixed points on X. **We assume that $\mathbb{W}$ is finite.**

Definition 25 (attracting set)**.**

For each $w \in \mathbb{W}$ we define the **attracting set**

$$X_w := \left\{ x \in X \mid \lim_{z \to 0} z \cdot x = w \right\}.$$

Since $w \in X_w$, we have that $\mathbb{C}^\times \cdot w = w$, so $\mathbb{C}^\times \curvearrowright T_w X$. But $\mathbb{C}^\times$-representations are just graded vector spaces, where

$$T_w X = \bigoplus_{n \in \mathbb{Z}} T_w X(n), \quad T_w X(n) := \{x \in T_w X \mid z \cdot x = z^n x \forall z \in \mathbb{C}^\times\}.$$

Note that $n = 0$ is not an eigenvalue since w is an isolated fixed point of the $\mathbb{C}^\times$-action, so locally around w the $\mathbb{C}^\times$-action always moves points nontrivially.

Let

$$T_w^+ X := \bigoplus_{n \in \mathbb{Z}_{>0}} T_w X(n).$$

Theorem 26 (Bialynicki-Birula)**.**

The action of $\mathbb{C}^\times$ on X decomposes X into a disjoint union of affine spaces.

1. The attracting sets form a decomposition $X = \bigsqcup_{w \in \mathbb{W}} X_w$.
2. We have natural isomorphisms $X_w \simeq T_w(X_w) \simeq T_w^+ X$, which commute with $\mathbb{C}^\times$-action.

Example 27.

Let $X = \mathbb{CP}^1$ with the standard $\mathbb{C}^\times$ action. Then $\mathbb{W} = \{0, \infty\}$. We have

$$X_0 = \mathbb{CP}^1 \setminus \{\infty\} \simeq \mathbb{A}^1,$$
$$X_\infty = \{\infty\} \simeq \mathbb{A}^0.$$

Then we have

$$\mathbb{CP}^1 = \mathbb{A}^1 \sqcup \mathbb{A}^0.$$

Remark 28.

There is a generalization of this result where $\mathbb{W}$ is not discrete; in this case, the pieces of the decomposition are parametrized by the connected components of $\mathbb{W}$.

There is an extensive relationship between the Bialynicki-Birula decomposition and Morse theory for Kähler manifolds, but we will skip it. We'll only stop to say that the Bialynicki-Birula decomposition on a Kähler manifold coincides with the cell decomposition from Morse theory.

2.5 Fixed Point Reduction

Let L be a Lie group.
Let X be a "reasonable" topological space; for example, this includes:

- a possibly singular closed complex subvariety of a complex manifold
- a finite-dimensional CW-complex
  and we have a continuous L-action on X.
  Let T be a compact torus contained in the center of L. Let X^T be the T-fixed points of X. Note that X^T is stable under the action of L.

  Proposition 29.
  Assume that L has finitely many connected components. Then:
  1. We have $[H^\bullet(X, \mathbb{C})] = [H^\bullet(X^T, \mathbb{C})] \in K^0(L)$.
  2. If $H^{odd}(X, \mathbb{C}) = 0$, then $H^{odd}(X^T, \mathbb{C}) = 0$.

Remark 30.

Statement (1) holds in $K^0(L-\text{mod})$, but since the action of L on cohomology actually factors through the group of connected components $\overline{L}$ (assumed to be finite here), then the equality also holds true in $K^0(\overline{L}-\text{mod})$. Furthermore, both Grothendieck groups can be upgraded to rings via the tensor product of representations.

The proof is very long so we'll skip it.

Corollary 31.

If the variety X has no odd-dimensional $\mathbb{Q}$-homology, then $[H^{even}(X)] = [H^{even}(X^T)] \in K^0(\overline{L})$.

2.6 Borel-Moore Homology

This will be the primary method of producing representations of various algebras and groups.

Once again, a space X will be a "reasonable" space, e.g. a locally compact topological space which has the homotopy type of a finite CW complex. Also, X is assumed to admit a closed embedding into a countable at infinity C^∞-manifold M. We assume that there is an open neighborhood $X \subset U \subset M$ such that X is a homotopy retract of U. A closed "subset" of a C^∞ manifold will mean a subset which has an open neighborhood for which it's a homotopy retract.

It is known that any complex or real algebraic variety satisfies these conditions so we assume most spaces are one of these.

Now we will provide equivalent definitions of the Borel-Moore homology. All coefficients are taken to be $\mathbb{C}$, but can be replaced with any field of characteristic 0.

Definition 32 (Borel-Moore homology)**.**

Let X be a "reasonable" space as indicated before. The **Borel-Moore homology** of X, denoted by $H_\bullet^{BM}(X)$, is one of the following equivalent definitions.

1) one-point compactification

Let $\widehat{X} = X \sqcup \{\infty\}$ be the one-point compactification of X. We define

$$H_\bullet^{BM}(X) := H_\bullet(\widehat{X}, \infty),$$

the relative homology of the pair $(\widehat{X}, \infty)$.

2) arbitrary compactification

Let $\overline{X}$ be an arbitrary compactification such that $(\overline{X}, \overline{X} \setminus X)$ is a CW-pair. Then

$$H_\bullet^{BM}(X) := H_\bullet(\overline{X}, \overline{X} \setminus X).$$

3) infinite singular chains

Let $C_\bullet^{BM}(X)$ be the chain complex of *infinite singular chains* $\sum_{i=0}^{\infty} a_i \sigma_i$, where σ_i is a singular simplex and $a_i \in \mathbb{C}$, but the sum is *locally finite*: for any compact set $D \subset X$, there are only finitely many nonzero a_i for which $D \cap \mathrm{supp}\sigma_i \neq \emptyset$. The usual boundary map ∂ is still valid on this chain complex because taking boundaries does not ruin the finiteness condition. Then

$$H_\bullet^{BM}(X) := H_\bullet(C_\bullet^{BM}(X), \partial).$$

4) Poincaré duality

Let M be a smooth oriented manifold of real dimension m, where $X \hookrightarrow M$ as a closed subset with an open neighborhood U for which X is a proper deformation retract (as above). Then

$$H_i^{BM}(X) \simeq H^{m-i}(M, M \setminus X).$$

In particular, for $X = M$, for any smooth (but not necessarily compact) variety M, we have a canonical isomorphism $H_i^{BM}(M) \simeq H^{m-i}(M)$.

This is an especially useful definition given the power of Poincaré duality.

There is another definition using the distribution deRham complex.

Remark 33.

If X is compact, then $H_\bullet^{BM}(X) = H_\bullet(X)$, for example using 3) infinite singular chains.

Proper pushforward

Let $f : X \to Y$ be a proper map. We define the proper pushforward map

$$f_* : H_\bullet^{BM}(X) \to H_\bullet^{BM}(Y).$$

We can define this using 1) one-point compactification. Extend f to a map $\widehat{f} : \widehat{X} \to \widehat{Y}$ with $\widehat{f}(\infty) = \infty$. Then we get an induced map

$$\widehat{f}_* : H_\bullet(\widehat{X}, \infty) \to H_\bullet(\widehat{Y}, \infty).$$

Long exact sequence

Let $U \subset X$ be open. Then for a compactification $\overline{X}$ of X (see 2) arbitrary compactification), we have an induced restriction map

$$H_\bullet^{BM}(X) = H_\bullet(\overline{X}, \overline{X} \setminus X) \to H_\bullet(\overline{X}, \overline{X} \setminus U) = H_\bullet^{BM}(U).$$

Now suppose we have a closed subset $V \subset X$. Then Let $U := X \setminus V$. We have

$$V \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} U.$$

Now embed $X \hookrightarrow M$ as a closed subset in a smooth manifold M. Poincaré duality gives us

$$\begin{aligned} H^{m-p}(M, M \setminus X) &\simeq H_p^{BM}(X), \\ H^{m-p}(M, M \setminus V) &\simeq H_p^{BM}(V). \end{aligned}$$

Since U is locally closed in M, we may shrink M to M' such that $U \subset M'$ is closed. Then excision implies that

$$H^{m-p}(M, M \setminus U) \simeq H^{m-p}(M', M' \setminus U) \simeq H_p^{BM}(U).$$

The long exact sequence in relative cohomology gives us:

$$\cdots \to H^k(M, M \setminus V) \to H^k(M, M \setminus X) \to H^k(M, M \setminus U) \to H^{k+1}(M, M \setminus V) \to \cdots$$

Now making the identifications, we find the **long exact sequence in Borel-Moore homology**

$$\cdots \to H_p^{BM}(V) \to H_p^{BM}(X) \to H_p^{BM}(U) \to H_{p-1}^{BM}(V) \to \cdots$$

Fundamental class

In ordinary homology, when M is a smooth compact oriented manifold, then there exists a well-defined fundamental class $[M] \in H_m(M)$, for $m = \dim_{\mathbb{R}} M$. Even when M is not compact, **there is a well-defined fundamental class in Borel-Moore homology**:

$$[M] \in H_m^{BM}(M).$$

This is particularly important and useful, and is an **essential feature** in Borel-Moore homology; the **fundamental class exists for any complex algebra variety, including those which are not smooth or compact**.

Example 34.

Let $M - \mathbb{R}^n$. Then $H_{>0}(M) = 0$, so there cannot be a fundamental class in ordinary homology. But in Borel-Moore homology, we can use the long exact sequence Long exact sequence of the pair $(S^n = \mathbb{R}^n \sqcup \infty, \mathbb{R}^n)$ to find that

$$H_i^{BM}(\mathbb{R}^n) = \begin{cases} \mathbb{C} \cdot \{[\mathbb{R}^n]\} & i = n, \\ 0 & i \neq n. \end{cases}$$

Let us now describe the fundamental class of an irreducible singular complex algebraic variety X. Let X^{reg} be the Zariski open dense subset of the nonsingular points of X. Let $m = \dim_{\mathbb{R}} X = \dim_{\mathbb{R}} X^{reg}$. Then X^{reg} has a canonical orientation coming from the complex structure on X^{reg}. Therefore we have a fundamental class

$$[X^{reg}] \in H_m^{BM}(X^{reg}).$$

Since $X \setminus X^{reg}$ has complex codimension at least 1, it follows that $\dim_{\mathbb{R}} X \setminus X^{reg} \leq m - 2$, so

$$H_{>m-2}^{BM}(X \setminus X^{reg}) = 0.$$

Then Long exact sequence implies that the inclusion $X^{reg} \hookrightarrow X$ induces a restriction map isomorphism

$$f : H_m^{BM}(X) \xrightarrow{\sim} H_m^{BM}(X^{reg}).$$

Following this, we define

$$[X] := f^{-1}([X^{reg}]),$$

so that the "fundamental class" of X in $H_m^{BM}(X)$ is defined to be the preimage of the fundamental class of X^{reg} in $H_m^{BM}(X^{reg})$.

If X is an arbitrary complex algebraic variety with irreducible components $X_1, \ldots, X_n$. Then we define

$$[X] := \sum_i [X_i].$$

Note that this is a non-homogeneous element!

Proposition 35.

Let X be a complex variety (not necessarily nonsingular) of complex dimension n. Let $X_1, \ldots, X_m$ be the n-dimensional irreducible components of X. Then the top Borel-Moore homology has a basis via $[X_i]$, i.e.,

$$H_{2n}^{BM}(X) = \bigoplus_{i=1}^{m} \mathbb{C} \cdot [X_i].$$

Intersection pairing

Let M be a smooth oriented manifold and Z, Z' two closed subsets (in the sense explained before). Our goal is to define an intersection pairing which consider cycles alongside their supports, rather than as just homology classes in the ambient manifold. We start with the standard cup product in relative cohomology:

$$\cup : H^{m-i}(M, M \setminus Z) \times H^{m-j}(M, M \setminus Z') \to H^{2m-j-i}\big(M, (M \setminus Z) \cup (M \setminus Z')\big).$$

The Poincaré dual of this gives us the **intersection pairing**

$$\cap : H_i^{BM}(Z) \times H_j^{BM}(Z') \to H_{i+j-m}^{BM}(Z \cap Z').$$

Remark 36.

This construction has a geometric meaning when M is a real analytic manifold and Z, Z' are closed analytic subsets in M. The basic idea is that we can identify $H_\bullet^{BM}(Z)$ with the homology of the complex formed by subanalytic chains, and then for subanalytic cycles $c \in H_\bullet^{BM}(Z)$ and $c' \in H_\bullet^{BM}(Z')$, we can choose representatives which intersect transversely at smooth points, hence the set-theoretic intersection $c \cap c'$ is a subanalytic cycle in $H_\bullet^{BM}(Z \cap Z')$. (We are skipping some steps, but this is the general idea.)

Intersection pairing with ordinary homology

Let M be a closed subset (in the sense explained at the beginning) of a smooth oriented manifold M. Then we have an analogue of Poincaré duality for **cohomology with compact support**:

$$H_c^{m-i}(M, M \setminus Z) \simeq H_i(Z).$$

We also have a cup product map

$$\cup : H_c^{m-i}(M, M \setminus Z) \times H^{m-j}(M, M \setminus Z') \to H_c^{2m-i-j}\left(M, (M \setminus Z) \cup (M \setminus Z')\right).$$

Now we apply Poincaré duality to obtain the **intersection pairing**

$$\cap : H_i(Z) \times H_j^{BM}(Z') \to H_{i+j-m}(Z \cap Z'),$$

where $m = \dim_{\mathbb{R}} M$.

In the special case when $Z = Z' = M$ and $i + j = m$ we get:

Proposition 37 (Poincaré duality)**.**

Assume M is an oriented connected (but not necessarily compact) smooth variety. Then for any j, the intersection pairing

$$\cap : H_j^{BM}(M) \times H_{m-j}(M) \to H_0(M) = \mathbb{C}$$

is non-degenerate. In particular,

$$H_j^{BM}(M) \simeq H_{m-j}(M)^* \simeq H^{m-j}(M).$$

Künneth formula

Let M_1, M_2 be arbitrary CW complexes and take compactifications $\overline{M_1}, \overline{M_2}$. The Künneth formula for ordinary homology is:

$$H_\bullet(\overline{M_1}, \overline{M_1} \setminus M_1) \otimes H_\bullet(\overline{M_2}, \overline{M_2} \setminus M_2) \simeq H_\bullet\left(\overline{M_1 \times M_2}, \overline{M_1 \times M_2} \setminus (\overline{M_1} \times M_2 \cup M_1 \times \overline{M_2})\right).$$

Then applying Poincaré duality we get a natural isomorphism

$$\boxtimes : H_\bullet^{BM}(M_1) \otimes H_\bullet^{BM}(M_2) \to H_\bullet^{BM}(M_1 \times M_2).$$

Restriction with supports

Let $i : N \hookrightarrow M$ be a closed embedding of oriented manifolds, with codimension d. Let $Z \subset M$ be a closed, possibly singular, subset. We define the **restriction with support in Z** functor

$$i^* : H_k^{BM}(Z) \to H_{k-d}^{BM}(Z \cap N), \quad c \mapsto c \cap [N].$$

Here, $c \cap [N]$ takes place in the ambient manifold M. This is the result of Poincaré duality on the standard restriction $i^* : H^\bullet(M, M \setminus Z) \to H^\bullet(N, N \setminus (N \cap Z))$.

Remark 38.

The map i^* crucially depends on the ambient manifold M, even though it is not explicitly present in the notation. For example, the map shifts the homology by $d = \dim M - \dim N$. So if we replace M by a really large smooth manifold then the shift will be even larger, and the map will even become zero if the ambient space becomes too large.

Diagonal reduction

Let M be a smooth oriented manifold and $i_\Delta : M_\Delta \hookrightarrow M \times M$ be the diagonal. Then for closed subsets $Z, Z' \subset M$ we have a set-theoretic equality

$$(Z \times Z') \cap M_\Delta = Z_\Delta \cap Z'_\Delta.$$

Similarly, for homology classes $c \in H_\bullet^{BM}(Z)$ and $c' \in H_\bullet^{BM}(Z')$, we have

$$c \cap c' = i_\Delta^*(c \boxtimes c') = (c \boxtimes c') \cap [M_\Delta].$$

In fact, we may even use this as an alternative definition of the Intersection pairing.

Smooth pullback

Let X be locally compact (not necessarily smooth) and $p : \widetilde{X} \to X$ a locally trivial fibration with smooth oriented fiber F of dimension d. We say that p is *oriented* if all transition functions of the fibration preserve the orientation of the fiber. If p is oriented, we can define a natural **pullback morphism**

$$p^* : H_\bullet^{BM}(X) \to H_{\bullet+d}^{BM}(\widetilde{X}).$$

We won't explicitly construct p^* here and will defer its description to chapter 8.

Now suppose X is embedded in a smooth oriented variety M, we have a locally trivial oriented fibration $\overline{p} : \widetilde{M} \to M$ with fiber F, and $p : \widetilde{X} := \overline{p}^{-1}(X) \to X$ is the restriction (which is also a fibration). In this case the pullback map is induced by the standard pullback morphism in cohomology $\overline{p}^* : H^\bullet(M, M \setminus X) \to H^\bullet(\widetilde{M}, \widetilde{M} \setminus \widetilde{X})$ via Poincaré duality. Furthermore:

Proposition 39 (projection formula).

Let $Z \subset M$ and $Z' \subset \widetilde{M}$ be closed subsets. Assume that $\overline{p}^{-1}(Z) \cap Z' \to M$ is proper. Write $Z \circ Z'$ for its image in M (it is a closed subset). Then for any $c \in H_\bullet^{BM}(Z)$ and $c' \in H_\bullet^{BM}(Z')$, we have

$$\overline{p}_*(p^* c \cap c') = c \cap (\overline{p}_* c) \in H_\bullet^{BM}(Z \cap Z').$$

Specialization in Borel-Moore homology

Let (S,o) be a smooth manifold of real dimension d with basepoint o. Write $S^\times := S \setminus \{o\}$. Given a (possibly singular) space Z and a map $\pi : Z \to S$, we set $Z_o := \pi^{-1}(o)$ and for any subset $S' \subset S$, write $Z(S') := \pi^{-1}(S')$. Assume that $\pi : Z(S^\times) \to S^\times$ is a locally trivial fibration (with possibly singular fibers; note that $\pi : Z \to S$ is *not* assumed to be locally trivial near o). Then we can define a **specialization map**

$$\lim_{s\to 0} : H_\bullet^{BM}(Z(S^\times)) \to H_{\bullet-d}^{BM}(Z_o).$$

The construction is as follows. Let $(B,o) \subset S$ be an open neighborhood diffeomorphic to $(\mathbb{R}^d, 0)$. Choose a decomposition $\mathbb{R}^d = \mathbb{R} \times \mathbb{R}^{d-1}$. Write $B_{>0}$ for the open subset corresponding to $\mathbb{R}_{>0} \times \mathbb{R}^{d-1}$, and shrink B as necessary until $\pi : Z(B_{>0}) \to B_{>0}$ is a trivial fibration with fiber F. Then the Künneth formula we find that

$$H_\bullet^{BM}(Z(B_{>0})) \xrightarrow{\sim} H_{\bullet-d}^{BM}(F) \otimes H_d^{BM}(B_{>0}) \xrightarrow{\sim} H_{\bullet-d}^{BM}(F) \otimes H_1^{BM}(\mathbb{R}_{>0}) \xrightarrow{\sim} H_{\bullet-d+1}^{BM}(Z(\mathbb{R}_{>0})).$$

Then we have three maps: first, we have the restriction map induced by $Z(B_{>0}) \hookrightarrow Z(S^\times)$; second, we have the chain of isomorphisms above, and third, we have the connecting homomorphism in the long exact sequence of the pair $Z(\mathbb{R}_{\geq 0}) = Z(\mathbb{R}_{>0}) \sqcup \mathbb{Z}_o$. We define the composition to be

$$\lim_{s\to 0} : H_\bullet^{BM}(Z(S^\times)) \to H_\bullet^{BM}(Z(B_{>0})) \xrightarrow{\sim} H_{\bullet-d+1}^{BM}(Z(\mathbb{R}_{>0})) \to H_{\bullet-d}^{BM}(Z_o).$$

It turns out that the specialization map does not depend on the choices made.

Furthermore, specialization enjoys a transitive property. Let $S_1 \subset S$ be a smooth submanifold of codimension k. Let $\varepsilon^* : H_\bullet^{BM}(Z(S^\times)) \to H_{\bullet-k}^{BM}(Z(S_1^\times))$ be the pullback map induced by the embedding $Z(S_1^\times) \hookrightarrow Z(S^\times)$. Then

Lemma 40.

Specialization is compatible with restriction:

$$\overset{S}{\lim_{s\to 0}} = \overset{S_1}{\lim_{s\to 0}} \circ \varepsilon^*.$$

Also, the intersection pairing commutes with specialization, in that

$$\cap \circ \lim_{s\to 0} = \lim_{s\to 0} \circ \cap.$$

Cohomology action

There is a natural $H^\bullet(Z)$-module structure on $H_\bullet^{BM}(Z)$. It is constructed as follows.

Choose a closed embedding $i : Z \hookrightarrow M$ into a C^∞-manifold M such that Z is a homotopy retract of M. Then the restriction map induced by i gives an isomorphism $i^* : H^\bullet(M) \xrightarrow{\sim} H^\bullet(Z)$. We also have the cap product $\cap : H^i(M) \times H^j(M, M \setminus Z) \to H^{i+j}(M, M \setminus Z)$. Taking the Poincaré dual of the last two terms yields the **cohomology action map**, sometimes denoted by $\cup$ product,

$$\cup : H^i(Z) \otimes H_k^{BM}(Z) \to H_{k-i}^{BM}(Z), \quad a \otimes c \mapsto a \cdot c.$$

One can check that this does not depend on the choice of i; this will also become clear in chapter 8 using the sheaf-theoretic definition of Borel-Moore homology.

We also have compatibility between the cohomology action map and the intersection pairing. Let Z, Z' be closed subsets in M. For $a \in H^\bullet(Z)$, write $a|_{Z\cap Z'}$ for the natural restriction to $H^\bullet(Z \cap Z')$ induced by $Z \cap Z' \hookrightarrow Z$. Let $c \in H_\bullet^{BM}(Z)$ and $c' \in H_\bullet^{BM}(Z')$. Then

$$(a \cdot c) \cap c' = a|_{Z\cap Z'} \cdot (c \cap c') \in H_\bullet^{BM}(Z \cap Z').$$

Thom isomorphism

Let $\pi : V \to X$ be a locally-trivial oriented C^∞-vector bundle of rank r. Then the **Euler class** $e(V) \in H^r(V)$. If V is a complex vector bundle then $e(V)$ is just the top Chern class of V.

Proposition 41 (Thom isomorphism)**.**

Let $i : X \hookrightarrow V$ be the zero section.

1. The pullback maps i^* and π^* induce mutually inverse isomorphisms of Borel-Moore homology:

$$H_\bullet^{BM}(X) \underset{\sim}{\overset{\sim}{\leftrightarrows}} H_{\bullet+r}^{BM}(V).$$

2. For any $c \in H_\bullet^{BM}(X)$, we have $i^* i_*(c) = e(V) \cdot c$.

Corollary 42.

Let N be an oriented closed submanifold of an oriented C^∞-manifold M, of (real) codimension d. Let $i : N \hookrightarrow M$ be the inclusion. Then for all $c \in H_\bullet^{BM}(N)$ we have $i^* i_*(c) = e(T_N M) \cdot c$.

Proof.
The basic idea is that we want an open tubular neighborhood $U \supset N$ in M diffeomorphic to $T_N M$. Then using some excision, we may replace (M, N) with $(T_N M, N)$. Then Proposition 40 (Thom isomorphism) part (2) yields the result.

We can also relate subbundles. □

Corollary 43.

Let $p : V \to Z$ be an oriented vector bundle and $W \subset V$ an oriented subbundle, with inclusion map $j : W \hookrightarrow V$. Then

$$j_*[W] = p^* \, e(V/W) \cdot [V] \in H_\bullet^{BM}(V).$$

Proof.
Proposition 40 (Thom isomorphism) part (1) implies that the statement is equivalent to applying j^* to it, as an equality in $H_\bullet^{BM}(W)$. Then Proposition 40 (Thom isomorphism) part (2) implies that

$$j^* j_*[W] = p^* \, e(V/W) \cdot [W],$$

while the RHS is

$$j^*(p^* \, e(V/W) \cdot [V]) = p^* \, e(V/W) \cdot j^*[V] = p^* \, e(V/W) \cdot [W],$$

since $j^*[V] = [W]$.

□

Access intersection formula

Let Z_1, Z_2 be closed oriented submanifolds of an oriented C^∞-manifold M. Our goal is to compute the intersection pairing $[Z_1] \cap [Z_2]$ in the most general case.

Proposition 44.

Assume that $Z := Z_1 \cap Z_2$ is "clean" in the sense that for all $z \in Z$,

$$T_z Z_1 \cap T_z Z_2 = T_z Z.$$

For example, this holds for any transverse intersection. Define the vector bundle on z

$$T_{1,2} := T_z M/(T_z Z_1 + T_z Z_2).$$

Then

$$[Z_1] \cap [Z_2] = e(T_{1,2}) \cdot [Z] \in H_\bullet^{BM}(Z),$$

where $[Z_i] \in H_\bullet^{BM}(Z_i)$ and $e(T_{1,2}) \in H^\bullet(Z)$.

2.7 Convolution in Borel-Moore Homology

Let M_1, M_2, M_3 be connected, oriented, C^∞-manifolds. Let

$$Z_{1,2} \subset M_1 \times M_2, \qquad Z_{2,3} \subset M_2 \times M_3$$

be closed subsets. Define

$$Z_{1,2} \circ Z_{2,3} := \{(m_1, m_3) \in M_1 \times M_3 \mid \exists m_2 \in M_2 \text{ s.t. } (m_1, m_2) \in Z_{1,2} \,\&\, (m_2, m_3) \in Z_{2,3}\}.$$

Example 45.

Let $f : M_1 \to M_2$ and $g : M_2 \to M_3$ be smooth maps. Then

$$\text{Graph}(f) \circ \text{Graph}(g) = \text{Graph}(g \circ f).$$

So we can think of $Z_{i,j}$ as multivalued maps $M_i \to M_j$, and thus $Z_{1,2} \circ Z_{2,3}$ can be thought of as the composition.

Let $p_{i,j} : M_1 \times M_2 \times M_3 \to M_i \times M_j$ be the projection maps. Assume that

$$p_{1,3} : p_{1,2}^{-1}(Z_{1,2}) \cap p_{2,3}^{-1}(Z_{2,3}) = Z_{1,2} \times_{M_2} Z_{2,3} \to M_1 \times M_3$$

is proper. Then $Z_{1,2} \circ Z_{2,3} = \text{im}(p_{1,3})$ defined above (restricted to the domain above). This is closed.

Definition 46 (convolution in Borel-Moore homology)**.**

Let $d := \dim_{\mathbb{R}} M_2$. We define the **convolution in Borel-Moore homology**

$$H_i^{BM}(Z_{1,2}) \times H_j(Z_{2,3}) \to H_{i+j-d}^{BM}(Z_{1,2} \circ Z_{2,3})$$

denoted

$$(c_{1,2}, c_{2,3}) \mapsto c_{1,2} * c_{2,3},$$

by translating the set-theoretic composition into composition of cycles. Specifically,

$$c_{1,2} * c_{2,3} = (p_{1,3})_* \left((c_{1,2} \boxtimes [M_3]) \cap ([M_1] \boxtimes c_{2,3})\right) \in H_\bullet^{BM}(Z_{1,2} \circ Z_{2,3}).$$

Note that $c_{1,2} \boxtimes [M_3] = p_{1,2}^* c_{1,2}$ and $[M_1] \boxtimes c_{2,3} = p_{2,3}^* c_{2,3}$. Also note that the direct image is well-defined since $p_{1,3}$ was assumed to be proper, so the support is indeed contained in a reasonable set.

Convolution is associative and compatible with lots of things, such as Künneth, specialization, base change, etc.