Let (M, ω) be a symplectic manifold. Recall that we have the exact sequence

$$0 \to \mathbb{C} \to \mathcal{O}(M) \to \mathfrak{X}_{symp}(M) \to H^1(M, \mathbb{R}) \to 0.$$

the first three terms are exact from our theory. Now note that $\xi \in \mathfrak{X}(M)$ is symplectic if and only if

$$\mathcal{L}_\xi \omega = 0 \iff d(i_\xi \omega) = 0 \iff i_\xi \omega \text{ is closed.}$$

Moreover, $i_\xi \omega = -df$ is exact if and only if it is in the image of $\mathcal{O}(M) \to \mathfrak{X}_{symp}(M)$. Prove allow us to complete the exact sequence.

Definition. Let G be a lie group acting on M. We say this action is **symplectic** if for all $g \in G$ we have $g^* \omega = \omega$. In other words, $\omega(g.\, x, g.\, y) = \omega(x, y)$.

Lemma. If G is a symplectic action on (M, ω), then the infinitesimal G-action gives a Lie algebra homomorphism

$$\mathfrak{g} := \text{Lie}(G) \longrightarrow \text{symplectic vector fields on } M.$$

Proof. We have that

$$L_X \omega = \lim_{t \to 0} \frac{d}{dt} \exp(tX)^* \omega = \lim_{t \to 0} \frac{d}{dt} \omega = 0.$$

□

Definition. A symplectic G-action is *Hamiltonian* if there is a Lie algebra homomorphisms $H : \mathfrak{g} \to \mathcal{O}(M)$, denoted $x \mapsto H_x$ which makes the triangle commute:

$$(BIG-TRIANGLE)$$

We can view $H : \mathfrak{g} \times M \to \mathbb{C}$. This allows us to define the *moment map* $\mu : M \to \mathfrak{g}^*$ by

$$\mu(m)(?) := H_?(m).$$

Lemma.

1. For any $x \in \mathfrak{g}$, we have $H_x = \mu^* x$.
2. The map $\mu^* : \mathbb{C}[\mathfrak{g}^*] \to \mathcal{O}(M)$ commutes with the Poisson structure.
3. If G is connected, the μ is G-equivariant relative to coadjoint action on $\mathfrak{g}^*.

Proof.
In (1.) we are viewing $x \in \mathfrak{g}$ as an element of $(\mathfrak{g}^*)^*$ in the natural way. Thus, we have

$$\mu^* x(m) = x(\mu(m)) = \mu(m)(x) = H_x(m).$$

For (2.), it suffices to prove on linear functions. Thus, for $x, y \in \mathfrak{g}$ we have that

$$\{\mu^* x, \mu^* y\} = \{H_x, H_y\} = H_{[x,y]} = \mu^*[x, y] = \mu^*\{x, y\}.$$

where $[x, y] = \{x, y\}$ in the last equality because of how the Poisson algebra structure is defined on $\mathbb{C}[\mathfrak{g}^*]$ (recall the last section) and the second equality is from the Lie algebra homomorphism structure of the Hamiltonian.

For (3.), since G is connected, it suffices to prove the infinitesimal equivariance.

Let $m \in M$ and $\lambda := \mu(m)$ where $\mu : M \to \mathfrak{g}^*$. Let $\mu_* : T_m M \to \mathfrak{g}^*$. We want to prove that

$$\mu_*(\xi_x) = \mathrm{ad}^* x(\lambda)$$

for all $x \in \mathfrak{g} = (\mathfrak{g}^*)^*$ and $m \in M$ where ξ_x is the vector field corresponding to x. To prove that this equation holds, we check that both sides have the same values after substituting $y \in \mathfrak{g}$ where we consider y a function on $\mathfrak{g}^*$. For the left hand side, we have

$$\langle y, \mu_*(\xi_x)\rangle = \mu_*(\xi_x)(y) = \xi_x(y \circ \mu) = \xi_x(\mu^* y) = \xi_{H_x}(\mu^* y) = \{H_x, \mu^* y\} = \{\mu^* x, \mu^* y\}.$$

For the right hand side, we have

$$\langle y, \mathrm{ad}^* x(\lambda)\rangle = \langle [x, y], \lambda\rangle = \lambda([x, y]) = \mu(m)([x, y]) = H_{[x,y]}(m) = \{H_x, H_y\}(m) = \{\mu^* x, \mu^* y\}(m).$$

This suffices for the proof. □

Example. Let $M = \mathbb{C}^2$ and $G = \mathrm{SL}_2(\mathbb{C})$. The lie algebra is $\mathfrak{sl}_2(\mathbb{C})$ which has basis

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The vector fields corresponding to e, f, h are

$$e \mapsto q\frac{\partial}{\partial p}, \quad f \mapsto p\frac{\partial}{\partial q}, \quad h \mapsto p\frac{\partial}{\partial p} - q\frac{\partial}{\partial q}.$$

Indeed, we show the computation for the first one:

$$\frac{d}{dt}|_{t=0} \exp\left(t\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 0 \end{bmatrix}.$$

After getting the vector fields, you can solve the relevant differential equations to get a valid Hamiltonian function

$$e \mapsto q^2/2, \quad f \mapsto -p^2/2, \quad h \mapsto pq.$$

Identifying $\mathfrak{sl}_2(\mathbb{C})^*$ and $\mathfrak{sl}_2(\mathbb{C})$ via the non-degenerate bilinear form $(A, B) \mapsto \mathrm{Tr}(A \cdot B)$, then we have the moment map $\mu : \mathbb{C}^2 \to \mathfrak{sl}_2(\mathbb{C})$ defined by

$$\mu(p,q) = \frac{1}{2}\begin{pmatrix} pq & -p^2 \\ q^2 & -pq \end{pmatrix}.$$

Note that the image is nilpotent. In Chapter 3, we see that this is a 2-fold covering of the nilpotent cone in $\mathfrak{sl}_2(\mathbb{C})$ ramified at the origin.

Example. Let $M = T^*X$ and let G act on X. Recall that we have Lie algebra homomorphims

$$\mathfrak{g} \to \mathfrak{X}(X) \to \mathfrak{X}(T^*X)$$

defined by $x \mapsto u_x \mapsto \tilde{u}_x$. In this case, we have $\pi_* \tilde{u} = u$. Here is a result from symplectic geometry:

Lemma. Let $f : M \to M$ be a diffeomorphism. Then $f^* : T^*M \to T^*M$ is a symplectomorphism.

In particular, if G acts on X, then G acts on T^*M via symplectomorphisms. Moreover, recall that we proved yesterday that

$$\widetilde{u} = \xi_{\lambda(\widetilde{u})}.$$

This immediately gives the following result:

Proposition. For any G-manifold X, the G action on T^*X is Hamiltonian with Hamiltonian

$$x \mapsto H_x = \lambda(\widetilde{u}_x) \in \mathcal{O}(T^*X).$$

Lemma. There is a natural G-equivariant isomorphism

$$T^*(G/P) \simeq G \times_P \mathfrak{p}^\perp.$$

Proof Sketch.

Let $\mathfrak{g}_{G/P} = G/P \times \mathfrak{g}$ be a trivial bundle. Consider the canonical vector bundle morphism

$$\mathfrak{g}_{G/P} \to T(G/P)$$

The fiber at each $x \in G/P$ is $\mathfrak{p}_x$ the stabilizer lie algebra. Since stabilizer groups are conjugate, the various $\mathfrak{p}_x$ are related via the adjoint action. Thus we have

$$T(G/P) \simeq G \times_P \mathfrak{g}/\mathfrak{p}.$$

Taking the dual, we get

$$T^*(G/P) \simeq G \times_P (\mathfrak{g}/\mathfrak{p})^* = G \times_P \mathfrak{p}^\perp.$$

Proposition. Under the isomorphism $T^*(G/P) \simeq G \times_P \mathfrak{p}^\perp$ the moment map μ is given explicitly by

$$(g, \alpha) \mapsto \mathrm{Ad}^*(g)\alpha, \quad g \in G, \alpha \in \mathfrak{p}^\perp.$$

Proof.

The moment map sends (g, α) to the linear function $\mu(g, \alpha) : \mathfrak{g} \to \mathbb{C}$ given by $x \mapsto H_x(g, \alpha)$, $x \in \mathfrak{g}$, where H_x is the Hamiltonian for x. Recall that $H_x = \lambda(\widetilde{x})$ where $\widetilde{x}$ is a lift of x. Also recall that (g, α) corresponds to $\mathrm{Ad}^*(g)\alpha$ as an element of $T^*(G/P)$. Then we have

$$\lambda(\widetilde{x})(g, \alpha) = \mathrm{Ad}^*(g)\alpha(\pi_*(\widetilde{x})) = \mathrm{Ad}^*(g)\alpha(x).$$

□

Section 1.5. Coisotropic subvarieties

Let (M, ω) be a symplectic manifold with Poisson bracket $\{\cdot, \cdot\}$ on $\mathcal{O}(M)$.

Recall: a subvariety $\Sigma \subseteq M$ is coisotropic if

$$(T_p\Sigma)^{\perp\omega} \subseteq T_p\Sigma.$$

Let $\mathcal{J}_\Sigma \subseteq \mathcal{O}(M)$ be the defining ideal of Σ.

Proposition. The subvariety Σ is coisotropic if and only if $\{\mathcal{J}_\Sigma, \mathcal{J}_\Sigma\} \subset \mathcal{J}_\Sigma$, that is, if and only if $\mathcal{J}_\Sigma$ is a Lie subalgebra, not necessarily ideal.

Proof Sketch.

Suppose that $\{\mathcal{J}_\Sigma, \mathcal{J}_\Sigma\} \subset \mathcal{J}_\Sigma$. This holds if and only if

$$f, g \in \mathcal{J}_\Sigma \implies \omega(\xi_f, \xi_g)(m) \equiv 0$$

for all $m \in \Sigma^{reg}$.

For any smooth point $m \in \Sigma^{reg}$ and $f \in \mathcal{J}_M$, $W = T_m\Sigma$, $V = T_mM$, we have that $df = 0$ on W so $df \in W^\perp$. This implies that $\xi_f \in W^{\perp\omega}$. But since $\mathcal{J}_\Sigma$ is the defining ideal, we know that $W^\perp$ is spanned by ξ_f. Thus

$$\omega(W^{\perp\omega}, W^{\perp\omega}) = 0.$$

This proves that $W^{\perp\omega}$ is isotropic so W is coisotropic. We can run this argument in reverse to get the if and only if. □

Let $\Sigma \subset M$ be a smooth coisotropic subvariety and $m \in \Sigma$. Then

1. The restriction of ω to $T_m\Sigma$ is degenerate (dimension counting argument)
2. $\mathrm{Rad}(\omega|_{T_m\Sigma}) = (T_m\Sigma)^{\perp\omega} \subset T_m\Sigma$.

If we put all of these radicals of ω at each fiber of $T_m\Sigma$, we get a vector subbundle

$$(T\Sigma)^{\perp\omega} \subset T\Sigma.$$

This vector bundle is actually *integrable.* That is,

Proposition. There exists a foliation on Σ such that for any $m \in \Sigma$, the space $(T_m\Sigma)^{\perp\omega}$ is equal to the tangent space at m to the leaf of the foliation.

Explanation: Thus, we can partition Σ into "leaves" (submanifolds) such that the tangent spaces in each submanifold corresponds to the tangent space $(T_m\Sigma)^{\perp\omega}$.

This directly follows from

Theorem. (Frobenius Integrability Theorem)
Let $E \subset T\Sigma$ be a vector subbundle of the tangent bundle on a manifold Σ. Then E is integrable if and only if sections of E form a Lie subalgebra.

So, in our case, since $\{\xi_f, f|_\Sigma = \text{constant}\}$ spans $(T_m\Sigma)^{\perp\omega}$, we want to prove that when $f|_\Sigma = constant$ and $g|_\Sigma = constant$, then we have

$$[\xi_f, \xi_g] = \xi_{\{f,g\}} \in (T\Sigma)^{\perp\omega}$$

since $\{f, g\}|_\Sigma = constant$ from the fact that the defining ideal is a subalgebra. $\square$

Example. Let M be symplectic and let $f \in \mathcal{O}(M)$. Let Σ be the zero variety of f. Suppose that df does not vanish on Σ. Thus Σ is a codimension 1 subvariety. It is coisotropic since (f) is a Lie subalgebra. The foliation which gives $(T\Sigma)^\perp$ is the foliation given by ξ_f and the integral curves that it traces.

Theorem. Let A be a solvable algebraic group with a Hamiltonian action on a symplectic algebraic variety M. Let $\mathfrak{a} = Lie(A)$ and let μ be the moment map

$$\mu : M \to \mathfrak{a}^*$$

Then for any coadjoint orbit $\mathbb{O} \subset \mathfrak{a}^*$ the set $\mu^{-1}(\mathbb{O})$ is either empty or is a coisotropic subvariety of M.

Proof Sketch.

Lemma. Let (V, ω) be a symplectic vector space. A vector subspace $\Sigma \subset V$ is coisotropic if and only if it contains a lagrangian subspace $\Lambda \subset \Sigma$.

Proof. It $\Lambda \subset \Sigma$ is lagrangian, then

$$\Sigma \supset \Lambda = \Lambda^{\perp\omega} \supset \Sigma^{\perp\omega}.$$

Conversely, if Σ is coisotropic. Then $\Sigma \supset \Sigma^{\perp\omega}$ and $\Sigma/\Sigma^{\perp\omega}$ is again symplectic. We can pick any lagrangian $\overline{\Lambda} \subseteq \Sigma/\Sigma^{\perp\omega}$. Its pullback to Σ will be a lagrangian subspace. $\square$

Lemma. (Technical, so we omit the proof)

For $N \subset M$ irreducible subvariety (M is smooth) and $f \in \mathcal{O}(N)$ a nonconstant regular function. For any $c \in \mathbb{C}$ we define $D_c = f^{-1}(c)$ and assume D_0 is nonempty. Then there is a Zariski-open dense subset $D_0^{gen} \subset D_0$ such that

D_0^{gen} is contained in the smooth locus of D_0 and for any point $x \in D_0^{gen}$, there is a sequence of complex numbers $c_1, c_2, \ldots \to 0$ and a sequence of points D_{c_i} such that

1. $x_i \to x$ (in Hausdorff topology) and x_i is a smooth point in D_{c_i}.
2. $T_{x_i} D_{c_i} \to T_x D_0$ where convergence takes place in the space of Grassmanians $(\dim N - 1)$ planes in TM.
3. The numbers $c_1, c_2, \ldots$ can be picked generically.

Let A be our solvable Lie group with lie algebra $\mathfrak{a}$. Then we have a codimension 1 normal subgroup $A_1 \subset A$ with lie algebra $\mathfrak{a}_1 \subset \mathfrak{a}$.

(Why is this true? Exercise: Prove that a solvable lie algebra has a codimension 1 ideal)

Claim. Consider the map

$$\mathbb{O} \hookrightarrow \mathfrak{a}^* \xrightarrow{p} \mathfrak{a}_1^*$$

where p is the pullback of the inclusion $\mathfrak{a}_1 \hookrightarrow \mathfrak{a}$. Then, I claim that we have one of the two alternatives:

1. $\dim p(\mathbb{O}) = \dim \mathbb{O}$ in which case $p(\mathbb{O})$ is a single A_1-orbit.
2. $\dim p(\mathbb{O}) < \dim \mathbb{O}$. In this case, the dimension of any A_1-orbit in $p(\mathbb{O})$ equals $\dim \mathbb{O} - 2$.

Note that since A_1 is a normal subgroup, we have a natural A action on $\mathfrak{a}_1$ and hence $\mathfrak{a}_1^*$. It is not difficult to see that $p : \mathfrak{a}^* \to \mathfrak{a}_1^*$ is A-equivariant. Thus $p(\mathbb{O})$ is a A-orbit in $\mathfrak{a}_1^*$.

Let $o \in p(\mathbb{O})$. Since $\dim \mathfrak{a}_1 = \dim \mathfrak{a} - 1$, if we consider the tangent vectors induced by $\mathfrak{a}_1$ at o via the A_1 action and the tangent vectors induced by $\mathfrak{a}$ at o via the A action, we must have

$$\dim(\mathfrak{a}_1 \cdot o) \geq \dim(\mathfrak{a} \cdot o) - 1.$$

(The tangent vectors from $\mathfrak{a}_1$ is missing at most one dimension)
Thus, we have

$$\dim(A_1 \cdot o) \geq \dim(A \cdot o) - 1 = \dim p(\mathbb{O}) - 1$$

where the equality follows since $p(\mathbb{O})$ is an A-orbit. When $\dim p(\mathbb{O}) = \dim \mathbb{O}$ we have

$$\dim \mathbb{O} \geq \dim(A_1 \cdot o) \geq \dim \mathbb{O} - 1.$$

All A_1 orbits in $p(\mathbb{O})$ are symplectic manifolds, so $A_1 \cdot o$ has even dimension. Since $\mathbb{O}$ is a coadjoint orbit of A, it is also symplectic and has even dimension. This implies that dimensions of A_1 orbits in $p(\mathbb{O})$ have dimension $\dim \mathbb{O} = \dim p(\mathbb{O})$, which means that $p(\mathbb{O})$ consists of only one A_1 orbit.

In the case where $\dim p(\mathbb{O}) < \dim \mathbb{O}$, we must have $\dim p(\mathbb{O}) = \dim \mathbb{O} - 1$. But then

$$\dim \mathbb{O} - 1 = \dim p(\mathbb{O}) \geq \dim(A_1 \cdot o) \geq \dim(A \cdot o) - 1 = \dim p(\mathbb{O}) - 1.$$

Since dimension of $A_1 \cdot o$ must be even, we have that it must be $\dim \mathbb{O} - 2$. □

Proof of Theorem.

Recall that $\mathbb{O} \subset \mathfrak{a}^*$ is a coadjoint orbit and we want to show that $\mu^{-1}(\mathbb{O})$ is a coisotropic subvariety.

We induct on $\dim A$. Let $A_1 \subset A$ be codimension 1 normal subgroup. Suppose that $\dim p(\mathbb{O}) = \dim \mathbb{O} - 1$. Then $\mathbb{O}$ is an open part of $p^{-1}(p(\mathbb{O}))$ (look at the co-dimension and it is preserved under pre-image). Thus, we want to prove that

$$\mu^{-1}(p^{-1}p(\mathbb{O})) = \mu_1^{-1}(p(\mathbb{O}))$$

is coisotropic. But this follows from induction since this is a union of coisotropic subvarieties the pre-images of coadjoint orbits in $\mathfrak{a}_1^*$ which are coisotropic from the induction.

Now suppose that $\dim p(\mathbb{O}) = \dim \mathbb{O}$. From our earlier argument, we know that $N = \mu^{-1}(p^{-1}p(\mathbb{O}))$ is coisotropic. We know that $\mathbb{O}$ is codimension 1 in $p^{-1}p(\mathbb{O})$.

We argue locally: Let P be a local equation of $\mathbb{O}$, i.e., a function on $p^{-1}p(\mathbb{O})$ such that $P \not\equiv 0$ and $P|_{\mathbb{O}} = 0$. This implies that

$$\mu^{-1}(\mathbb{O}) = N \cap \{\mu^* P = 0\}.$$

Let $f = \mu^* P$. Since we are working locally, we can assume that N is irreducible and f does not vanish on N. Let

$$\Sigma_c := N \cap \{f = c\}$$

Lemma. For generic $c \in \mathbb{C}$, Σ_c is coisotropic.

Our Goal: Show Σ_0 is coisotropic. From the Lemma before, we can find $x_i \to x \in \Sigma_0$ such that the lemma holds. Then

$$T_{x_i}\Sigma_{c_i} \to T_{x_0}\Sigma_0.$$

From the lemma, there are lagrangian subspaces $\Lambda_i \subset T_{x_i}\Sigma_{c_i}$. We can pick a subsequence i_k such that $\Lambda_{i_k} \to \Lambda \subset T_{x_0}\Sigma_0$ (Grassmanians are compact). Then Λ is isotropic since all Λ_{i_k} are lagrangian. By looking at dimension, this actually implies Λ is lagrangian. This completes the proof.