Section 1: Symplectic Manifolds

In this section, X will be

1. a C^∞-manifold
2. complex manifold
3. algebraic variety over $\mathbb{C}$

Definition. (Symplectic Structure)
A symplectic structure on X is a non-degenerate regular (smooth, holomorphic, algebraic) 2-form ω such that $d\omega = 0$.

Example 1. Let $X = \mathbb{C}^{2n}$ with coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$. Then

$$\omega = dx_1 \wedge dy_1 + \ldots + dx_n \wedge dy_n$$

is a symplectic structure.

Now, we give more examples of symplectic structures.

Example 2. Let M be any manifold. Then the cotangent bundle has a canonical symplectic structure.

Idea: we construct a one-form λ on T^*M and set $\omega = d\lambda$.
To construct λ, for every $\alpha \in T^*M$, we want to associate a linear function $\lambda_\alpha : T_\alpha(T^*M) \to 0$.
Let $\xi \in T_\alpha(T^*M)$ and $\pi : T^*M \to M$ be the natural projection. Then, we define

$$\lambda_\alpha(\xi) := \langle \alpha, \pi_* \xi \rangle \in \mathbb{C}.$$

Using Coordinates

Suppose $y_1, \ldots, y_n$ is a coordinate system for M and $x_1, \ldots, x_n$ the dual coordinate system. These give a chart in T^*M. Any $\xi \in T_\alpha(T^*M)$ can be written in the form

$$\xi = \sum b_i \frac{\partial}{\partial x_i} + \sum c_i \frac{\partial}{\partial y_i}.$$

We have that

$$\pi_*(\xi) = \sum c_i \frac{\partial}{\partial y_i}.$$

Thus, we see that

$$\lambda_\alpha(\xi) = \langle \alpha, \pi_* \xi \rangle = \sum c_i \alpha \left(\frac{\partial}{\partial y_i} \right) = \sum c_i x_i(\alpha).$$

(The last step is how the dual coordinates are constructed)
We can write this as

$$\lambda = \sum x_i dy_i \implies d\lambda = \sum dx_i \wedge dy_i.$$

Thus $d\lambda$ is locally of the form in Example 1, hence non-degenerate.

Example 3. Let G be a Lie group with lie algebra $\mathfrak{g}$. The adjoint G-action on $\mathfrak{g}$ gives gives rise to a transposed *co-adjoint* G-action on $\mathfrak{g}^*$

Proposition. Any co-adjoint orbit $\mathbb{O} \subseteq \mathfrak{g}^*$ has a natural symplectic structure.

Proof.
Pick any point $\alpha \in \mathbb{O} \subseteq \mathfrak{g}^*$. We want a skew-symmetric form on $T_\alpha \mathbb{O}$. We have a natural isomorphism $\mathbb{O} \simeq G/G^\alpha$ where G^α is the isotropy group of α. So we can write

$$T_\alpha \mathbb{O} \simeq \mathfrak{g}/\mathfrak{g}^\alpha.$$

We want a skew-symmetric form on $\mathfrak{g}/\mathfrak{g}^\alpha$. We first define a form

$$\omega_\alpha : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}.$$

given by $\omega_\alpha(x, y) = \alpha([x, y])$. To show that this descends to a non-degenerate form on $\mathfrak{g}/\mathfrak{g}^\alpha$, it is enough to show that $\mathfrak{g}^\alpha$ is the radical of ω_α. Indeed, we know that $g \in G^\alpha$ if and only if $\mathrm{Ad}^* g(\alpha) = \alpha$. Differentiating, $x \in \mathfrak{g}^\alpha$ if and only if $\mathrm{ad}^*(x)(\alpha) = 0$. This right condition is equivalent to for all $y \in \mathfrak{g}$

$$0 = \langle \mathrm{ad}^*(x)\alpha, y \rangle = \langle \alpha, \mathrm{ad}(x)(y) \rangle = \langle \alpha, [x, y] \rangle = \alpha([x, y]).$$

Thus $\omega_\alpha : \mathfrak{g}/\mathfrak{g}^\alpha \times \mathfrak{g}/\mathfrak{g}^\alpha \to \mathbb{C}$ is non-degenerate skew-symmetric on $\mathbb{O}$. It is enough to prove that ω is closed.

Recall Cartan's formula for exterior derivative: Given ξ_1, ξ_2, ξ_3 vector fields we have

$$\begin{aligned} d\omega(\xi_1, \xi_2, \xi_3) = {} & \xi_1 \cdot \omega(\xi_2, \xi_3) + \xi_3 \cdot \omega(\xi_1, \xi_2) + \xi_2 \cdot \omega(\xi_3, \xi_1) \\ & - (\omega([\xi_1, \xi_2], \xi_3) + \omega([\xi_3, \xi_1], \xi_2) + \omega([\xi_2, \xi_3], \xi_1)). \end{aligned}$$

Any element $x \in \mathfrak{g}$ gives rise to a vector field ξ_x on $\mathbb{O}$. In general, if G acts on M, then every vector in $\mathfrak{g}$ gives a vector field in M via

$$x \in \mathfrak{g} \longrightarrow \xi_x(m) := \frac{d}{dt}|_{t=0} \exp(xt).\, m$$

In our case, since $\mathbb{O}$ is homogeneous, the tangent spaces of $\mathbb{O}$ are spanned by the ξ_x. Explicitly in our case, the vector field ξ_x is given by

$$\xi_x(\alpha) = \frac{d}{dt}|_{t=0} \mathrm{Ad}^*(\exp(tx))\alpha.$$

Hence, it suffices to prove that $d\omega(\xi_x, \xi_y, \xi_z) = 0$ for all $x, y, z \in \mathfrak{g}$. From a well-known formula of Cartan for the exterior derivative, we have

$$\begin{aligned}(d\omega)(\xi_1, \xi_2, \xi_3) = \xi_1 \cdot \omega(\xi_2, \xi_3) + \xi_3 \cdot \omega(\xi_1, \xi_2) + \xi_2 \cdot \omega(\xi_3, \xi_1) \\ - (\omega([\xi_1, \xi_2], \xi_3) + \omega([\xi_3, \xi_1], \xi_2) + \omega([\xi_2, \xi_3], \xi_1)).\end{aligned}$$

It is not difficult to see that $\omega(\xi_y, \xi_z)(\alpha) = \alpha([y, z])$. We also have

$$\begin{aligned}\xi_x \omega_\alpha(\xi_y, \xi_z) &= \xi_x \alpha([y, z]) \\ &= \frac{d}{dt}|_{t=0} \mathrm{Ad}^*(\exp(tx))\alpha([y, z]) \\ &= \frac{d}{dt}|_{t=0} \alpha(\mathrm{Ad}(\exp(tx))[y, z]) \\ &= \alpha(\mathrm{ad}(x)[y, z]) \\ &= \alpha([x, [y, z]]).\end{aligned}$$

Then ω is closed from the Jacobi identity.

Section 2: Poisson Algebras

In this section, A be a commutative $\mathbb{C}$ algebra.

Definition. A commutative algebra A endowed with an additional $\mathbb{C}$-bilinear anti-symmetric bracket $\{\cdot, \cdot\} : A \times A \to A$ is called a *Poisson algebra* if the following hold:

1. A is a Lie algebra with respect to $\{\cdot, \cdot\}$;
2. Leibniz rule: $\{a, b \cdot c\} = \{a, b\} \cdot c + b \cdot \{a, c\}$ for all $a, b, c \in A$.
   The bracket $\{\cdot, \cdot\}$ is called a *Poisson bracket* and we say that it gives a *Poisson structure* to A.

Any symplectic manifold has a natural Poisson algebra.

Let (M, ω) be a symplectic manifold. The non-degenerate 2-form gives a canonical isomorphism $TM \simeq T^*M$. Thus, we can define a $\mathbb{C}$-linear map $\xi : \mathcal{O}(M) \to \mathfrak{X}(M)$ denoted $f \mapsto \xi_f$ such that

$$\omega(?, \xi_f) = df$$

or equivalently $i_{\xi_f}\omega = -df$. For any vector field η and any function f, we have

$$\omega(\eta, \xi_f) = \eta f.$$

We define a bracket on $\mathcal{O}(M)$ in the following way:

$$\{f, g\} = \omega(\xi_f, \xi_g) = \xi_f g = -\xi_g f.$$

In fact, the vector fields ξ_f also preserve the symplectic structure.

Definition. A vector field ξ is called *symplectic* if it preserves the symplectic form, i.e., $L_\xi \omega = 0$.

Lemma. For any $f \in \mathcal{O}(M)$, ξ_f is symplectic.

Proof.
From Cartan's homotopy formula, we have

$$L_{\xi_f}\omega = i_{\xi_f}d\omega + d(i_{\xi_f}\omega) = 0 + 0 = 0.$$

$\square$

Proposition. The assignment $f \mapsto \xi_f$ gives a bracket preserving map

$$(\mathcal{O}(M), \{\cdot, \cdot\}) \longrightarrow (\text{symplectic vector fields on } M, [\cdot, \cdot])$$

Proof. We want to show that $[\xi_f, \xi_g] = \xi_{\{f,g\}}$. It is a general fact about the lie derivative that

$$\xi \cdot \omega(\xi_1, \xi_2) = L_\xi(\omega(\xi_1, \xi_2)) = (L_\xi\omega)(\xi_1, \xi_2) + \omega(L_\xi\xi_1, \xi_2) + \omega(\xi_1, L_\xi\xi_2).$$

Thus, when ξ is symplectic we have

$$\xi \cdot \omega(\xi_1, \xi_2) = \omega([\xi, \xi_1], \xi_2) + \omega(\xi_1, [\xi, \xi_2]).$$

This implies that for any vector field η, we get that

$$\begin{aligned} \xi_f \cdot \omega(\xi_g, \eta) &= \omega([\xi_f, \xi_g], \eta) + \omega(\xi_g, [\xi_f, \eta]) \\ &= \omega([\xi_f, \xi_g], \eta) - \xi_f\eta g + \eta\xi_f g. \end{aligned}$$

Since the right hand side is $-\xi_f\eta g$ which implies that

$$\omega(\eta, [\xi_f, \xi_g]) = \eta\{f, g\} \implies \xi_{\{f,g\}} = [\xi_f, \xi_g].$$

$\square$

Theorem. The algebra $\mathcal{O}(M)$ of regular functions on a symplectic manifold M together with $\{\cdot, \cdot\}$ is a Poisson algebra.

Proof. To prove that the Jacobi identity, we know that

$$[\xi_f, \xi_g]h = \xi_{\{f,g\}}h = \{\{f, g\}, h\}.$$

On the other hand,

$$[\xi_f, \xi_g]h = \xi_f\xi_g h - \xi_g\xi_f h = \xi_f\{h, g\} - \xi_g\{f, h\} = \{\{h, g\}, f\} - \{g, \{f, h\}\}.$$

This proves the Jacobi identity. The Leibniz rule is obvious since differentiation along vector fields are derivation. $\square$

Section 3: Poisson Structures arising from Non-commutative Algebras

Let B be an associative filtered (non-commutative) algebra with unit. In other words, we have a filtration

$$\mathbb{C} = B_0 \subset B_1 \subset B_2 \subset \ldots, \quad \bigcup_i B_i = B$$

such that $B_i \cdot B_j \subseteq B_{i+j}$ for all $i, j \geq 0$. Let $A = \mathrm{gr}B = \bigoplus_i (B_i/B_{i-1})$. The multiplication in B gives a well-defined product

$$B_i/B_{i-1} \otimes B_j/B_{j-1} \to B_{i+j}/B_{i+j-1}$$

making A into an associative algebra.

Definition. We say B is *almost commutative* if $\mathrm{gr}B$ is commutative with respect to the above product.

Proposition. If B is almost commutative, then $\mathrm{gr}\ B$ has a natural Poisson structure.

Proof.
We first define a map

$$\{\cdot, \cdot\} : B_i/B_{i-1} \times B_j/B_{j-1} \to B_{i+j-1}/B_{i+j-2}$$

by doing the following. Let $a_i \in B_i/B_{i-1}$ and $a_j \in B_j/B_{j-1}$ and let $b_i \in B_i, b_j \in B_j$ be representatives. Then, we let

$$\{a_1, a_2\} = (b_i b_j - b_j b_i) \pmod{B_{i+j-2}}.$$

We know that $b_i b_j - b_j b_i \in B_{i+j-1}$ from almost commutativity. It is not hard to check that this is independent of our choice of representative. We leave it as an exercise to check the remaining axioms. □

Example. Let B be the associative $\mathbb{C}$-algebra with generators $p_1, \ldots, p_n$, $q_1, \ldots, q_n$ and relations

$$[p_i, p_j] = 0 = [q_i, q_j] \text{ and } [p_i, q_j] = \delta_{ij}.$$

One way to realize this in a natural way is to consider

$$\mathrm{Diff} = \left\{ \sum a_{\underline{k}}(x) \frac{\partial^{k_1 + \ldots + k_n}}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}}, a_{\underline{k}}(x) \in \mathbb{C}[x_1, \ldots, x_n], \underline{k} = (k_1, \ldots, k_n) \right\}$$

an algebra of polynomial differential operators. This is isomorphic to B via the assignment $p_i \leftrightarrow \frac{\partial}{\partial x_i}$ and $q_i \leftrightarrow x_i$.

We can give another construction of the same algebra in a coordinate free way. Let (V, ω) be a symplectic vector space and c a dummy central variable. Note that there is a basis $p_1, \ldots, p_n$, $q_1, \ldots, q_n$ such that

$$\omega(p_i, p_j) = 0 = \omega(q_i, q_j) \text{ and } \omega(p_i, q_j) = \delta_{ij}.$$

Form the algebra $TV \otimes \mathbb{C}[c]$ and give TV and $\mathbb{C}[c]$ the standard gradings by assigning c and every element $v \in V$ grading 1. Give $TV \otimes \mathbb{C}[c]$ the natural grading extending this one. Let

$$\widetilde{B} = TV \otimes \mathbb{C}[c]/(v_1 \otimes v_2 - v_2 \otimes v_1 - c \cdot \omega(v_1, v_2)).$$

This has a filtration F_k which consists of all monomials of degree $\leq k$ in the generators written in any order. We also have

$$\mathrm{gr}_F \widetilde{B} = S(V)[c] = \mathbb{C}[p_1, \ldots, p_n, q_1, \ldots, q_n, c]$$

because of the defining relation in $\widetilde{B}$. This implies that $\widetilde{B}$ is almost commutative, hence the polynomial ring $\mathbb{C}[p_1, \ldots, p_n, q_1, \ldots, q_n, c]$ has a Poisson algebra structure. Explicitly, this is given by

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right) \cdot c$$

for $f, g \in \mathbb{C}[p_1, \ldots, p_n, q_1, \ldots, q_n, c]$.

Proof Idea: It is easy to see that both sides satisfy the Leibniz rule. Thus, it is enough to show that they agree on generators. $\square$

By specializing $c = 1$, we get a Poisson bracket on $SV = \mathbb{C}[p_1, \ldots, p_n, q_1, \ldots, q_n]$. We can identify $SV \simeq \mathbb{C}[V^*]$. Since ω is non-degenerate, we have a canonical isomorphism $V^* \simeq V$. Thus, we can give V^* a symplectic structure and SV would be the algebra of polynomial functions on V^*. The elements $p_1, \ldots, p_n, q_1, \ldots, q_n \in V$ would form a coordinate system of V where the symplectic form on V^* is of the form

$$\omega|_{V^*} = \sum dp_i \wedge dq_i.$$

Observation: The Poisson algebra structure given by the symplectic structure on V^* is exactly the Poisson algebra structure that we just defined.

If $f, g \in SV$ are homogeneous elements of degree 2, then from our formula we know that $\{f, g\}$ is of degree 2 as well. Thus, the Poisson bracket defines a bracket on S^2V, the space of degree 2 homogeneous elements.

Lemma 1.3.5. The elements of degree 2 form a Lie algebra isomorphic canonically to $\mathfrak{sp}(V)$, the symplectic Lie algebra.

Recall that the symplectic lie algebra $\mathfrak{sp}(V)$ consists of all morphisms $A : V \to V$ which preserve the symplectic form: $\omega(Ax, y) + \omega(x, Ay) = 0$.

Proof. If $f \in S^2V$ and $g \in S^1V$, then $\deg\{f, g\} = 1$. Thus via the Poisson bracket, S^2V acts on S^1V. This gives a map $S^2V \to \mathrm{End}(S^1V) = \mathrm{End}(V)$.

For $f, g \in S^1V$, we have that $\omega(f,g) = \{f,g\}$ from construction. Thus, for $h \in S^2V, f, g \in S^1V$, we have

$$\omega(\{h,f\},g) + \omega(f,\{h,g\}) = \{\{h,f\},g\} + \{f,\{h,g\}\} = \{h,\omega(f,g)\} = 0$$

from the Jacobi identity. This shows that we actually have a Lie algebra morphism $S^2V \to \mathfrak{sp}(V)$. To prove that this is isomorphism, it is enough to check that they have the same dimension and that it is injective. This is injective because if $f \in S^2V$ was in the kernel, that would imply that f would commute with S^1V. Since S^1V generates SV, the algebra SV would have non-trivial center. But it is clear from the formula for the lie bracket that this algebra has no center. $\square$

Example: Algebra of Regular Differential Operators

The results in this example hold for X a smooth manifold, open subset of $\mathbb{C}^n$, or a smooth complex affine algebraic variety.

Let $\mathcal{T}(X)$ be the vector space of regular vector fields on X, and let $\mathcal{D}(X)$ be the sub-algebra of $\mathrm{End}_{\mathbb{C}}\mathcal{O}(X)$ generated by $\mathcal{O}(X)$ and $\mathcal{T}(X)$ where $\mathcal{O}(X)$ is viewed as a operator by multiplication. Then we have a filtration

$$\mathcal{O}(X) = \mathcal{D}_0(X) \subset \mathcal{D}_1(X) \subset \mathcal{D}_2(X) \subset \dots$$

where $\mathcal{D}_1(X) = \mathcal{O}(X) + \mathcal{T}(X)$ and $\mathcal{D}_n = \mathcal{D}_1(X) \cdot \mathcal{D}_{n-1}(X)$. Let $\mathcal{D}(X) = \bigcup D_n(X)$ be the algebra of regular differential operators.

When X is a smooth manifold or open subset of $\mathbb{C}^d$, then locally we can write

$$u = \sum_{n_1,\dots,n_d} u_{n_1,\dots,n_d}(x)\partial_1^{n_1}\dots\partial_d^{n_d}, \quad u_{n_1,\dots,n_d} \in \mathcal{O}(X).$$

Moreover, in the case where X is a smooth manifold, using partitions of unity we can prove for any $u : \mathcal{O}(X) \to \mathcal{O}(X)$ if on any chart it restricts to a operator of the above form, then $u \in \mathcal{D}(X)$.

Principal Symbols

We now define a way to associate a polynomial function to every differential operator.

First consider the setting $X \subseteq \mathbb{C}^n$. Let $u \in \mathcal{D}(X)$ be a differential operator of order n. Then u can be written in the above form. Let $x_1, \dots, x_d, p_1, \dots, p_d$ be the standard coordinates on T^*X. We define the *principal symbol* $\sigma_n(u)$ by

$$\sigma_n(u) = \sum_{n_1+\dots+n_d=n} u_{n_1,\dots,n_d}(x) \cdot p_1^{n_1}\dots p_d^{n_d} \in \mathcal{O}(T^*X).$$

When X is a smooth manifold, this can be defined in any local chart.

The principal symbol of first order can be defined intrinsically. First let $u = \xi + f$ be an order 1 differential operator where $\xi \in \mathcal{T}(X)$ and $f \in \mathcal{O}(X)$. Then $\sigma_1(u) = \sigma_1(\xi)$. Let $\xi = \sum u_i\partial_i$ and let

$\alpha \in T_x^* X$. Then

$$\sigma_1(\xi)(\alpha) = \sum u_i(x) p_i(\alpha) = \sum u_i(x) \alpha(\partial_i) = \langle \alpha, \xi_x \rangle.$$

We have written σ_1 in an intrinsic way.

Inspired by this, on a smooth manifold X there is a well-defined regular function $\sigma_n(u)$ on T^*X which restricts to the previously defined one in any local chart. From the intrinsic definition of $\sigma_1(\xi)(\alpha) = \langle \xi_{\pi\alpha}, \alpha \rangle$, we see that σ_1 is a regular function on T^*M. To prove that σ_n can be extended to a regular function, note that any $u \in \mathcal{D}_n(X)$ can be written as the sum of monomials of the form $\xi_1 \cdot \ldots \cdot \xi_r$ for $r \leq n$. On a local chart, it is not difficult to verify that

$$\sigma_n(\xi_1 \ldots \xi_r) = \sigma_1(\xi_1)\sigma_1(\xi_2) \ldots \sigma_1(\xi_r)$$

when $r = n$ and 0 otherwise. So we get a coordinate-free expression for σ_n which proves that σ_n is a well-defined regular function.

We can do something similar when X is algebraic, but we omit this construction. See page 31 if interested.

The upshot is we get a well-defined morphism

$$\sigma_n : \mathcal{D}_n(X)/\mathcal{D}_{n-1}(X) \longrightarrow \text{degree } n \text{ homogeneous polynomials on } T^*X.$$

In all three settings, σ_n ends up being an isomorphism. For references on where to find proof, see page 32. Putting all of these together, we get an algebraic isomorphism

$$\text{gr } \mathcal{D}(X) \longrightarrow \bigoplus_{n \geq 0} \text{homogeneous polynomial functions on } T^*X \text{ of degree } n = \mathcal{O}_{pol}(T^*X).$$

Since $\mathcal{D}(X)$ is almost commutative, there is a canonical Poisson structure on $\text{gr } \mathcal{D}(X)$ which carries over to $\mathcal{O}_{pol}(T^*X)$ via the isomorphism. It turns out that this is the same Poisson structure given by the symplectic structure on T^*X.

Theorem. The Poisson structure on $\mathcal{O}_{pol}(T^*X)$ is the same as the one arising from the Poisson structure on T^*X.

Proof. Since both Poisson structures satisfy the Leibniz rule, it suffices to prove that they are the same on generators. We work locally and verify that they are the same by explicit calculation. Given two vector fields $u, v \in \mathcal{T}(X)$, we have

$$u = \sum u_i(x) \frac{\partial}{\partial x_i}, \quad v = \sum v_i(x) \frac{\partial}{\partial x_i}.$$

Then, we have

$$\sigma_1(u) = \sum u_i(x) p_i, \quad \sigma_1(v) = \sum v_i(x) p_i.$$

We want to prove that $\sigma([u,v]) = \{\sigma_1(u), \sigma_1(v)\}$. We can compute

$$[u,v] = \sum_{i,j} \left(u_i \frac{\partial v_j}{\partial x_i} \frac{\partial}{\partial x_j} - v_j \frac{\partial u_i}{\partial x_j} \frac{\partial}{\partial x_i} \right)$$

which gives

$$\sigma_1([u,v]) = \sum_{i,j} \left(u_i \frac{\partial v_j}{\partial x_i} p_j - v_j \frac{\partial u_i}{\partial x_j} p_i \right).$$

On the other hand, from the formula is the earlier example we have

$$\begin{aligned} \{\sigma_1(u), \sigma_1(v)\} &= \sum_k \left(\frac{\partial \sigma_1(u)}{\partial p_k} \frac{\partial \sigma_1(v)}{\partial x_k} - \frac{\partial \sigma_1(v)}{\partial p_k} \frac{\partial \sigma_1(u)}{\partial x_k} \right) \\ &= \sum_{i,j} \left(u_i \frac{\partial v_j}{\partial x_i} p_j - v_j \frac{\partial u_i}{\partial x_j} p_i \right) = \sigma_1([u,v]). \end{aligned}$$

□

To a vector field u on X there is a canonical way to associate a vector field $\tilde{u}$ on T^*X.

Indeed, we have that a vector field on X gives an infinitesimal diffeomorphism (flow) on X, which gives an infinitesimal diffeomorphism of T^*X which then gives a vector field on T^*X.

In the algebraic setting, we will construct this lift explicitly. Note that we have a map

$$\tilde{u} : \mathcal{T}(X) + \mathcal{O}(X) \to \mathcal{T}(X) + \mathcal{O}(X)$$

defined by $\tilde{u}(\xi + f) = [u, \xi] + u(f)$ where u acts by the Lie derivative. When $\mathcal{T}(X)$ is a free $\mathcal{O}(X)$ module, then $\mathcal{O}(T^*X) \simeq S\mathcal{T}(X)$. A vector field on T^*X is equivalent to a derivation on $\mathcal{O}(T^*X)$. Since $\tilde{u}$ can be extended to a derivation on $S\mathcal{T}(X)$, this is a well-defined vector field.

In the general case, we can cover X by locally free open sets, define it on each set, and then glue them together since the definition is coordinate free. By our construction,

$$\pi_*(\tilde{u}_\alpha) = u_x \text{ where } \pi : T^*X \to X.$$

Over manifolds, we can make the construction explicit as follows:

Let u be a vector field on X
let $\pi : T^*X \to X$ be the canonical projection.

Let $f_u : T^*X \to X$ be the function defined by

$$f_u(\xi) = \xi(u_{\pi(\xi)}).$$

Let $\tilde{u} = X_{f_u}$, the symplectic vector field on T^*X. Then

1. $\tilde{u}$ is a lift of u
2. $L_{\tilde{u}}\lambda = 0$.

Claim. $\widetilde{u}$ is a symplectic vector field on T^*X.
Proof Sketch. Verify that λ is invariant under the infinitesimal automorphism on T^*X induced from infinitesimal automorphism on X. This means that $L_{\widetilde{u}}\lambda = 0$. Thus, we have

$$L_{\widetilde{u}}\omega = L_{\widetilde{u}}d\lambda = dL_{\widetilde{u}}\lambda = 0.$$

□

Recall that any function on T^*X gives a symplectic vector field on T^*X. In fact, given a vector field u, it turns out that $\sigma_1(u) \in \mathcal{O}(T^*X)$ exactly gives the vector field $\widetilde{u}$ on T^*X.

Lemma. $\widetilde{u} = \xi_{\sigma_1(u)}$ and $\sigma_1(u) = \lambda_\bullet(\widetilde{u}) = i_{\widetilde{u}}\lambda$.

Proof. Note that

$$0 = L_{\widetilde{u}}\lambda = i_{\widetilde{u}}d\lambda + di_{\widetilde{u}}\lambda = i_{\widetilde{u}}\omega + d(i_{\widetilde{u}}\lambda)$$

Thus, we have that $\omega(\cdot, \widetilde{u}) = d(i_{\widetilde{u}}\lambda)$. We can then compute for any $\alpha \in T^*X$,

$$(i_{\widetilde{u}}\lambda)(\alpha) = \lambda_\alpha(\widetilde{u}) = \alpha(\pi_*(\widetilde{u})) = \alpha(u) = \langle \alpha, u \rangle = \sigma_1(u)(\alpha).$$

Thus $\sigma_1(u) = \lambda_\bullet(\widetilde{u})$. This completes the proof to both parts. □

Example. (Poisson structure coming from a finite dimensional Lie algebra)

Let $\mathfrak{g}$ be a finite dimensional Lie algebra. The universal enveloping algebra $U\mathfrak{g}$ has a canonical filtration

$$\mathbb{C} = U_0\mathfrak{g} \subset U_1\mathfrak{g} \subset \dots$$

where $U_k\mathfrak{g}$ is the $\mathbb{C}$ span of monomials of degree $\leq j$ formed by elements in $\mathfrak{g}$.

Theorem. (PBW Theorem)
There are canonical graded algebra isomorphisms

$$\mathrm{gr}\, U\mathfrak{g} \simeq S\mathfrak{g} = \mathbb{C}[\mathfrak{g}^*].$$

Thus, $U\mathfrak{g}$ is almost commutative and $\mathbb{C}[\mathfrak{g}^*]$ has a canonical Poisson structure. We describe this structure.

Let $e_1, \dots, e_n$ be a basis of $\mathfrak{g}$ and let c_{ij}^k be constants such that

$$[e_i, e_j] = \sum c_{ij}^k e_k.$$

Note that $\mathfrak{g} \simeq (\mathfrak{g}^*)^*$ so we can view elements in $\mathfrak{g}$ as linear functions on $\mathfrak{g}^*$. Let $x_1, \dots, x_n \in (\mathfrak{g}^*)^*$ be the coordinate functions corresponding to $e_1, \dots, e_n$ (e.g. $x_1(\alpha) = \alpha(e_1)$).

Proposition. One has the following two expressions for the Poisson bracket $\{f, g\}$ for $f, g \in \mathbb{C}[\mathfrak{g}^*]$.

$$\{f, g\} = \sum c_{ij}^k \cdot x_k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}.$$

Proof. This follows our standard argument. Both sides satisfy the Leibniz rule, it suffices to check both on linear functions. Note that we have $\{x, y\} = [x, y]$ and in particular $\{e_i, e_j\} = [e_i, e_j] = \sum_k c_{ij}^k e_k$. This should be enough. $\square$

We connect the Poisson structure of $\mathbb{C}[\mathfrak{g}^*]$ to the symplectic structure on coadjoint orbits in $\mathfrak{g}^*$. Note that any $f, g \in \mathbb{C}[\mathfrak{g}^*]$ can be viewed as regular functions on any $\mathbb{O} \subseteq \mathfrak{g}^*$. Thus, we can take the Poisson bracket with respect to the symplectic structure coming from $\mathbb{O}$. It turns out that this is the same as the bracket we just defined.

Proposition. For any regular functions $f, g \in \mathbb{C}[\mathfrak{g}^*]$ and any coadjoint orbit $\mathbb{O} \subseteq \mathfrak{g}^*$, we have

$$\{f, g\}|_{\mathbb{O}} = \{f|_{\mathbb{O}}, g|_{\mathbb{O}}\}_{\text{symplectic}}.$$

Proof. From the standard argument, we only need to show for linear functions. Let $x, y \in \mathfrak{g} = (\mathfrak{g}^*)^*$. Then $\{x, y\} = [x, y]$. For any $\alpha \in \mathbb{O}$, we have

$$\{x, y\}(\alpha) = [x, y](\alpha) = \alpha([x, y]) = \omega_\alpha(\xi_x, \xi_y) = \{x|_{\mathbb{O}}, y|_{\mathbb{O}}\}_{symplectic}.$$

Isotropic, Coisotropic, Lagrangian subvarieties

Let (V, ω) be a symplectic vector space.

Definition. A linear subspace $W \subset V$ is called

1. (Isotropic) If $W \subset W^{\perp\omega}$
2. (Co-isotropic) If $W \supset W^{\perp\omega}$.
3. (Lagrangian) If $W = W^{\perp\omega}$.

Example. Let $V = \mathbb{C}^{2n}$ and $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ a basis satisfying $\omega(e_i, e_j) = 0 = \omega(f_i, f_j)$ and $\omega(e_i, f_j) = \delta_{ij}$. Then

1. $W = \langle e_1, \ldots, e_k \rangle$ is isotropic
2. $W^{\perp\omega} = \langle e_1, \ldots, e_n, f_{k+1}, \ldots, f_n \rangle$ is coisotropic.
3. $\langle e_1, \ldots, e_n \rangle$ and $\langle f_1, \ldots, f_n \rangle$ are lagrangian.

Let (M, ω) be a symplectic manifold.

Definition. A subvariety Z of M is called an isotropic (coisotropic, langrangian) subvariety of M, if for any smooth point $z \in Z$, $T_z Z$ is an isotropic (coisotropic, lagrangian) subspace of $T_z M$.

Example. Let X be any manifold, and $M = T^*X$ its cotangent bundle with canonical symplectic form ω. Let $f \in \mathcal{O}(X)$. Then the image of $df : M \to T^*X$ is a lagrangian subvariety of T^*X. In fact, the image of a 1-form is lagrangian if and only if it is closed.

Proof sketch. Let $\eta : X \to T^*X$ be a 1-form. Then

$$(\eta^*\lambda)_p(v) = \lambda_{\eta(p)}(d\eta_p(v)) = \eta(p)(d\pi_{\eta(p)} \circ d\eta_p(v)) = \eta(p)(v).$$

Thus $\eta^*\lambda = \eta$ for all 1 forms. Since $\dim \eta(X) = \frac{1}{2}\dim T^*X$, it suffices to show that $\eta(X)$ is isotropic. In other words, we want $\eta^*\omega = 0$. But we can compute

$$\eta^*\omega = \eta^* d\lambda = d\eta^*\lambda = d\eta.$$

Thus it is lagrangian if and only if η is closed. $\square$

Definition. Let X be a manifold and T^*X its cotangent bundle. Given a submanifold $Y \subset X$, define $T^*_Y X$ to be the conormal bundle of Y. Each fiber will be

$$(T^*_Y X)_y = (T_y Y)^\perp \subset T^*_y X.$$

Proposition. The total space of the bundle $T^*_Y X$ is a lagrangian submanifold of T^*X stable under dilations along the fibers of T^*X.

Proof Sketch. Verify that $\dim T^*_Y X = 1/2 \cdot \dim T^*X$. Then we want to prove that $T^*_Y X$ is isotropic, that is $\omega|_{T^*_Y X} = 0$. It is enough to show that $\lambda|_{T^*_Y X} = 0$, but this follows from the definition of λ and $T^*_Y X$. $\square$

We say a subvariety of T^*X stable under dilations along the fibers is a *cone subvariety* of T^*X. Thus in the previous proposition, we showed that the conormal bundle is a lagrangian cone subvariety.

We can give a sort of converse and characterize lagrangian cone subvarieties in a cotangent bundle in terms of conormal bundles.

Let Eu be the Euler vector field generating the $\mathbb{C}^*$ action along the fibers of T^*X. In local coordinates, we have $\lambda = \sum p_i dq_i$, $Eu = \sum p_i \frac{\partial}{\partial p_i}$ and $\omega = \sum dp_i \wedge dq_i$ so we have that $i_{Eu}\omega = \lambda$
.

Lemma. (Kashiwara) Let X be a smooth algebraic variety. Assume $\Lambda \subset T^*X$ is a closed irreducible algebraic lagrangian subvariety. Let Y be the smooth part of $\pi(\Lambda)$ where $\pi : T^*X \to X$ is the projection. Then $\Lambda = \overline{T^*_Y X}$.

Proof Sketch.
Since Λ is $\mathbb{C}^*$-stable, Eu is tangent to Λ. Since it is lagrangian, for any vector ξ tangent to Eu we have

$$0 = \omega(Eu, \xi) = \lambda(\xi).$$

Thus $\lambda|_\Lambda = 0$. So if we pick any $\alpha \in \Lambda^{reg}$ such that $y = \pi(\alpha) \in Y$, by definition of λ we know that α vanishes on the image of $\pi_* : T_\alpha \Lambda \to T_y Y$. From Bertini-Sard's lemma, there is a Zariski open dense subset $\Lambda^{generic} \subset \Lambda^{reg}$ such that this map is surjective at any point. Hence $\alpha(T_y Y) = 0$ and $\alpha \in T_Y^* X$. This shows that $\Lambda^{generic} \subset T_Y^* X$. Both of these are irreducible varieties (since Λ is irreducible) of the same dimension since Λ is lagraigian, thus we have

$$\Lambda = \overline{\Lambda^{generic}} = \overline{T_Y^* X}.$$

□

Here is an application of this characterization:

Let V be a finite dimensional vector space, and let $G \subset \mathrm{PGL}(V)$ be an (irreducible) algebraic subgroup.

Theorem. Assume that G has finitely any orbits on $\mathbb{P}(V)$. There is a natural bijection between the G-orbits on $\mathbb{P}(V)$ and the G-orbits on $\mathbb{P}(V^*)$.

Proof Sketch.

Let $\tilde{G}$ be the inverse image of G under $\mathrm{GL}(V) \to \mathrm{PGL}(V)$. Then $\tilde{G}$ contains the scalars and it is enough to exhibit a natural bijection between $\tilde{G}$ orbits on V and $\tilde{G}$ orbits on V^*.

Note that we have canonical isomorphisms $T^*V = V \times V^* = T^*(V^*)$. Let p_V, p_{V^*} be the projections of the two factors. We now describe the correspondence between orbits.

Let $\mathbb{O} \subset V^*$ be a $\tilde{G}$ orbit. Then $T^*_{\mathbb{O}}(V^*)$ is a lagrangian cone subvariety of $V \times V^*$. Let

$$\tilde{\mathbb{O}} = \overline{p_V(T^*_{\mathbb{O}}(V^*))}.$$

Few observations: $\tilde{\mathbb{O}}$ is G-stable (look at fibers) and irreducible subvariety of V. We prove

1. $\tilde{\mathbb{O}}$ is the closure of a single $\tilde{G}$-orbit $\mathbb{O}^\vee \subset V$.
2. The orbit $\mathbb{O}$ can be recovered from the orbit $\mathbb{O}^\vee$.

Lemma. Let G be a connected algebraic group acting on an algebraic variety X. Then any irreducible G-stable algebraic subvariety of X is the closure of a single G-orbit.

Proof.

Let Y be this subvariety. Let $\mathbb{O}$ be the orbit of maximal dimension. Since $\mathbb{O}$ is not contained in the closure of any other orbit and there are only finitely many orbits, $\mathbb{O}$ must be an open subset of Y. Thus $\overline{\mathbb{O}} = Y$. □

This implies that $\tilde{\mathbb{O}}$ is the closure of a single $\tilde{G}$-orbit $\mathbb{O}^\vee \subset V$. To prove (2.), we can view $\overline{T^*_{\mathbb{O}}(V^*)}$ as an irreducible $\mathbb{C}^*$-stable lagrangian subvariety of T^*V. Thus, by a previous lemma, we have $\overline{T^*_{\mathbb{O}}(V^*)} = \overline{T^*_Y V}$ where Y is the smooth points of the image of $\overline{T^*_{\mathbb{O}}(V^*)}$ under the projection $p_V : V \times V^* \to V$. But this image is $\overline{\mathbb{O}}$. Thus $Y = \mathbb{O}^\vee$ and

$$\overline{T^*_{\mathbb{O}}(V^*)} = \overline{T^*_{\mathbb{O}^\vee}(V)}.$$

By switching $\mathbb{O}$ and $\mathbb{O}^\vee$ we get our bijection. □