Why Hopf algebras?

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1 Introduction

1.1 Purpose of this note

This is a really short note explaining the "purpose" of Hopf algebras. At least when I learned them, the definition of Hopf algebras was rather obtuse, and I couldn't quite figure out what they did. The topic of Hopf algebras came up in a discussion with Hunter Dinkins, which is when I found out that the notion of Hopf algebras was very natural, contrary to my prior belief. Hopefully this short note can pass this wisdom along to other people who are similarly confused by the rather complicated definition of Hopf algebras.

1.2 Layout

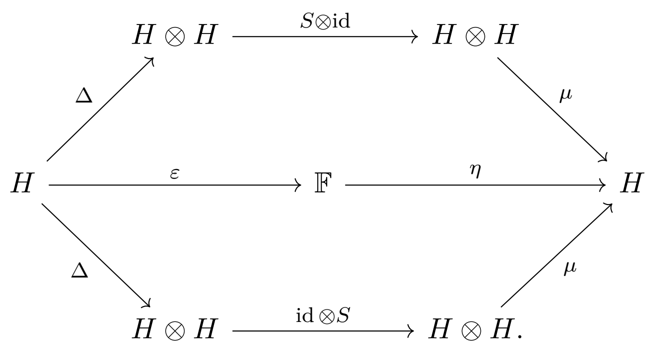
In the remainder of this section I'll briefly give an overview of the definition of a Hopf algebra, as well as some common examples. In §2 I'll give the motivation for why this notion is actually completely natural.

1.3 The definition

Definition 1.3.1. Fix a field $\mathbb{F}$. A **Hopf algebra** H over the field $\mathbb{F}$ is an $\mathbb{F}$-vector space H, equipped with maps:

- the **multiplication** map $\mu : H \otimes_{\mathbb{F}} H \to H$,
- the **unit** map $\eta : \mathbb{F} \to H$,
- the **comultiplication** map $\Delta : H \to H \otimes_{\mathbb{F}} H$,
- the **counit** map $\varepsilon : H \to \mathbb{F}$,
- the **antipode** map $S : H \to H$, which is an $\mathbb{F}$-linear anti-automorphism of H,

and these satisfy numerous relations. Briefly: the multiplication and unit maps turn H into an $\mathbb{F}$-algebra; the comultiplication and counit maps turn H into an $\mathbb{F}$-coalgebra, and the antipode map "intertwines" these two structures by making the following diagram commute:



These are all a bit complicated, so for the convenience of the reader, I'll briefly discuss some interpretations of these maps.

1.4 Algebra structure

First and foremost, H should be an $\mathbb{F}$-algebra. The multiplication map can be interpreted as any multiplication operation on a ring. The unit map is essentially saying that there's a copy of $\mathbb{F}$ inside (the center of) H. The relations here are the standard ones for μ and η to give H the structure of an $\mathbb{F}$-algebra.

1.5 Coalgebra structure

Formally, Δ and ε just play the (categorical) dual role of μ and η: for any diagram that μ and η satisfy, then Δ and ε satisfy the same diagram but with all of the arrows reversed.

1.6 Antipode

There are many ways to try to interpret the antipode map. One characterization is using Sweedler's notation: suppose $\Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)}$. Then the antipode satisfies the relation $S(c_{(1)})c_{(2)} = s_{(1)}S(c_{(2)}) = \varepsilon(c)1$.

Another characterization is using convolutions of maps $H \to H$. Generally speaking, let's suppose we have a coalgebra C and an algebra A. Then to two linear maps $f, g : C \to A$, we define their convolution $f \star g$ to be the composition

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{\mu} A.$$

Then the antipode is the unique linear map $S : H \to H$ for which $S \star \mathrm{id}_H = \mathrm{id}_H \star S = \eta \circ \varepsilon$ as maps $H \to H$.

1.7 Examples

Example 1.7.1 (group algebra)**.** Let $H = \mathbb{F}G$ be the group algebra of a group G.

- **(a)** The comultiplication is $\Delta(g) = g \otimes g$ for $g \in G$.
- **(b)** The counit is $\varepsilon(g) = 1$ for $g \in G$.
- **(c)** The antipode is $S(g) = g^{-1}$ for $g \in G$.

Example 1.7.2 (coordinate ring of group scheme)**.** Let $H = \mathbb{F}[G]$ be the ring of regular functions on a group scheme G.

- **(a)** The comultiplication is $\Delta(f)(x, y) = f(xy)$ for $f \in \mathbb{F}[G]$ and $x, y \in G$.
- **(b)** The counit is $\varepsilon(f) = f(1_G)$.
- **(c)** The antipode is $S(f)(x) = f(x^{-1})$.

Example 1.7.3 (tensor algebra)**.** Fix a vector space V. Let H be either the tensor algebra $T(V)$, the symmetric algebra $\mathbb{S}(V)$, or the exterior algebra $\Lambda(V)$.

- **(a)** The comultiplication is $\Delta(1) = 1 \otimes 1$, and $\Delta(v) = v \otimes 1 + 1 \otimes v$ for $v \in V$.
- **(b)** The counit is $\varepsilon(v) = 0$ for $v \in V$.
- **(c)** The antipode is $S(v) = -v$ for $v \in V$.

Example 1.7.4 (universal enveloping algebra)**.** Let $H = \mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of a Lie algebra $\mathfrak{g}$.

- **(a)** The comultiplication is $\Delta(X) = X \otimes 1 + 1 \otimes X$ for $X \in \mathfrak{g}$.
- **(b)** The counit is $\varepsilon(X) = 0$ for $X \in \mathfrak{g}$.
- **(c)** The antipode is $S(X) = -X$ for $X \in \mathfrak{g}$.

2 Motivation

So far, the above definitions probably aren't very satisfying. The most reasonable motivation I've seen for the creation of these notions is from group schemes and representations.

2.1 Motivation from group schemes

If G is a group scheme, and suppose for simplicity it's affine. Then the multiplication map on G is a map $G \times G \to G$, which is actually just the same as an $\mathbb{F}$-algebra map $\mathbb{F}[G] \to \mathbb{F}[G] \otimes \mathbb{F}[G]$. Notice that this is *not* the multiplication map on $\mathbb{F}[G]$, as the arrow is going the wrong way! It will turn out that $\mathbb{F}[G]$ is a Hopf algebra, and the comultiplication map is precisely the ring analogue of the multiplication map $G \times G \to G$. There are many natural maps on the group scheme G which correspondingly give us the Hopf algebra maps on $\mathbb{F}[G]$, we'll summarize it in the following table.

Scheme map on G	Hopf algebra map on $\mathbb{F}[G]$
multiplication $G \times G \to G$	comultiplication $\Delta : \mathbb{F}[G] \to \mathbb{F}[G] \otimes_{\mathbb{F}} \mathbb{F}[G]$
inclusion of identity $\operatorname{Spec} \mathbb{F} \to G$	counit $\varepsilon : \mathbb{F}[G] \to \mathbb{F}$
diagonal $G \to G \times G$	multiplication $\mu : \mathbb{F}[G] \otimes_{\mathbb{F}} \mathbb{F}[G] \to \mathbb{F}[G]$
structure map $G \to \operatorname{Spec} \mathbb{F}$	unit $\eta : \mathbb{F} \to \mathbb{F}[G]$
inverse $G \to G$	antipode $S : \mathbb{F}[G] \to \mathbb{F}[G]$

2.2 Motivation from representation theory

The other major motivation I've seen (thanks to Hunter Dinkins!) is for a reasonable notion of the category of representations. What does this mean? Well, one of fundamental constructions in representation theory is that we can do stuff like take the tensor product of representations and the dual of representations. Any time we have some nice algebra and want to work with its representations, we should be able to take tensor products and take duals. (Note that representations for the most common objects, groups and Lie algebras, are actually just representations of the corresponding algebras, the group algebra of the group and the universal enveloping algebra of the Lie algebra.)

Example 2.2.1. Let G be a group. Then for two G-representations V, W we can take their tensor product $V \otimes_{\mathbb{F}} W$, and we define the G-action on some element $v \otimes w \in V \otimes W$ as $g(v \otimes w) = gv \otimes gw$. Similarly, to a representation V, we can take the dual space $V^\vee$ and give it a G-module structure, by saying that for $\varphi \in V^\vee$, then $(g\varphi)(v) = \varphi(g^{-1}v)$.

Example 2.2.2. Let $\mathfrak{g}$ be a Lie algebra. Then for two $\mathfrak{g}$-modules V, W we can take their tensor product $V \otimes_{\mathbb{F}} W$ and give it a $\mathfrak{g}$-module structure: we have that $X(v \otimes w) = Xv \otimes w + v \otimes Xw$. Similarly, to a representation V, we can take the dual space $V^\vee$ and give it a $\mathfrak{g}$-module structure, by saying that for $X \in \mathfrak{g}$, $\varphi \in V^\vee$, and $v \in V$, that $(X\varphi)(v) = -\varphi(Xv)$.

Let's say A is an algebra, and we want to consider representations of it. What information do we need to take tensor products and duals of A-representations? Well, for A-modules V and W, then A doesn't canonically act on $V \otimes W$, *but* $A \otimes A$ *does*! Therefore **a comultiplication map $\Delta : A \to A \otimes A$ allows us to give the tensor product an A-module structure**. Similarly, to give an A-module structure to the dual space $V^\vee$, we need some A-anti-homomorphism to "reverse" the structure of A, and **that's exactly where the antipode comes in**: for $\varphi \in V^\vee$, $v \in V$, and $a \in A$, we just set $(a \cdot \varphi)(v) := \varphi(S(a) \cdot v)$. Thus to really get a nice category of representations, A should actually be a Hopf algebra!

Remark 2.2.3. If we want double dual to be identity - for example, in the category of finite-dimensional G-modules for a group G, or BGG Category $\mathcal{O}$ - we'd need $S^2 = \mathrm{id}$. There are several results which give sufficient conditions for this to be true; for example, the Larson-Radford theorem says that for a finite-dimensional Hopf algebra H over a field of characteristic 0, then $S^2 = \mathrm{id}$ iff H is semisimple iff $H^\vee$ is semisimple iff $\mathrm{tr}(S^2) \neq 0$. Another example is that if H is commutative or cocommutative, then $S^2 = \mathrm{id}$. Generally speaking, however, S^2 doesn't have to be the identity.