Differential Graded (Lie) Algebras, Derived Algebraic Geometry, and Deformation Theory

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Contents

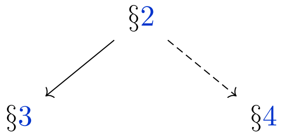
1 Introduction

These notes were written for two talks in a learning seminar on deformation theory. One of the goals of this learning seminar is to understand how differential graded Lie algebras control the theory of deformations, ultimately following [Man05]. Along the way, we learn a little bit about differential graded algebras and derived algebraic geometry.

These notes are mainly based on the first three sections of [EP21] and section 4 of [ABCW15].

1.1 Roadmap

In §2 we introduce the differential graded algebras and briefly delve into derived schemes. In §3 we give an overview on how the model structure on the category of differential graded algebras allows us to do things such as compute derived functors and understand derived schemes geometrically. In §4, we discuss how differential graded Lie algebras are deeply related to deformation theory. The sections §3 and §4 are independent of each other, and to be honest §4 does not seriously rely on any of the previous sections anyway, so that the flow chart looks something like:



2 Differential graded algebras

The main idea behind derived algebraic geometry is that derived stuff just works better. Endowing rings of functions with extra structure such as chain complexes gives us more flexibility and structure to work with, and certain families begin to behave better as well. The best formulation is probably using simplicial rings, but it ends up being the same as commutative differential graded algebras in characteristic 0. Another similar theory is spectral algebraic geometry, based on commutative ring spectra.

In this section we'll take a look at differential graded algebras and how they develop the theory of derived algebraic geometry.

We fix a base ring k which is a $\mathbb{Q}$-algebra; therefore, **if k is a field, it has characteristic** 0.

2.1 dg-algebras and affine dg-schemes

The main object of interest is:

Definition 2.1.1. A **differential graded k-algebra** (also known as dga or dg-algebra) is a chain complex $A = A_\bullet$ of k-modules along with:

- an associative k-linear multiplication $- \cdot - : A_i \times A_j \to A_{i+j}$ for all i, j,

- a unit $1 \in A_0$,
- and a k-linear differential $\delta : A_i \to A_{i-1}$ for all i, satisfying $\delta^2 = 0$ and $\delta(a \cdot b) = \delta(a) \cdot b + (-1)^{\deg a} a \cdot \delta(b)$.

We can represent a dga A by

$$\cdots \leftarrow A_{-3} \leftarrow A_{-2} \leftarrow A_{-1} \leftarrow A_0 \leftarrow A_1 \leftarrow A_2 \leftarrow A_3 \leftarrow \cdots .$$

A graded algebra A is **graded-commutative** if $a \cdot b = (-1)^{(\deg a) \cdot (\deg b)} b \cdot a$ for all homogeneous a, b. A dga which is also graded-commutative is called **differential graded-commutative algebra**, or cdga for short.

We can view a dga A in two ways: one way is to view it as a chain complex of k-modules equipped with a multiplication map compatible with the differential δ, and another way is to view it as a graded k-algebra equipped with a differential δ which is compatible with the multiplication structure.

We generally focus on dga which are concentrated in nonnegative degree, i.e. $A_i = 0$ for all $i < 0$, and we'll make that convention here (mainly for the reason that (co)homology in negative degrees are not so important in much of algebraic geometry). As a remark, conventionally algebraic geometers like cochain complexes, but we will use chain complexes here since it is more similar to simplicial objects.

Our goal is to define the category of k-cdga. We want to mimic the construction of the category of k-algebras, from which we can take the opposite category to define the category of affine k-schemes; this will serve as the blueprint to defining affine k-dg-schemes.

Definition 2.1.2. Any dga A is a chain complex, so **homology** $H_i(A_\bullet)$ makes sense. If A is a cdga, then $H_\bullet(A_\bullet)$ is itself a cdga, but the differential will be the 0 map since the elements are by definition killed by the differential.

Remark 2.1.3. You've likely seen lots of cdgas already: cohomology rings in algebraic topology are the primary suspects. The complexes which compute the (co)homology groups already have a natural multiplication (which is graded-commutative) on them which induces the (graded-commutative) multiplication in the (co)homology rings. Since we are dealing with the (co)homology rings, the differential is just zero and usually dropped from discussion. (Note that depending on if we're dealing with chain complexes of cochain complexes, the cdgas in question will be homologically or cohomologically graded; however this doesn't really change anything we've said.)

Example 2.1.4. The deRham complex of differential forms is a straightforward example. The deRham complex has a natural graded-commutative multiplication which is wedge product of two differential forms, and it respects the differential. The wedge product induces a multiplication on the cohomology ring $H^\bullet_{dR}(X)$ which is graded-commutative, and the differential is just 0. Hence we have two quasi-isomorphic cdgas: $H^\bullet_{dR}$ with the zero differential is quasi-isomorphic to the deRham complex with the differential given by the one on differential forms and with wedge product as multiplication).

Example 2.1.5. The complex of singular cochains naturally comes with a multiplication given by the Whitney cup product. This map respects the differential, hence turns the complex of singular cochains into a cdga. The Whitney cup product induces the cup product as a graded-commutative multiplication on the graded ring $H^\bullet(X)$, hence we have a quasi-isomorphism of cdgas between the cochain complex of singular cochains (with differential given by differential of singular cochains and multiplication by the Whitney cup product) and the singular cohomology ring $H^\bullet(X)$ (with zero differential and multiplication given by cup product).

Definition 2.1.6. A **morphism of dg-algebras** is a map $f : A_\bullet \to B_\bullet$ which respects the differentials and multiplication. Concretely, $f \circ \delta_A = \delta_B \circ f$ and $f(a \cdot_A b) = f(a) \cdot_B f(b)$. The usual notion of **quasi-isomorphism** (also known as **weak equivalence**) is the same: it is a morphism of dg-algebras which induces isomorphisms on all homology groups. Two cdga $A_\bullet$ and $B_\bullet$ are **quasi-isomorphic** if we have a "roof" diagram (working exactly as in derived categories) $A_\bullet \leftarrow C_\bullet \to B_\bullet$ of quasi-isomorphisms.

Definition 2.1.7. A dga A is **discrete** if it is concentrated in degree 0, i.e. $A_i = 0$ for all $i \neq 0$.

To any k-algebra A, we associate to it a discrete dga which is just A in degree 0 (and all other terms are 0).

Example 2.1.8. Let $M_\bullet$ be a graded k-module. The **free graded-commutative k-algebra** generated by $M_\bullet$ is

$$k[M_\bullet] \coloneqq (\mathrm{Sym}^\bullet(M_{\mathrm{even}})) \otimes_k (\Lambda^\bullet(M_{\mathrm{odd}})),$$

where the degree of a tensor of homogeneous elements is just the sum of the degrees of the elements. To make this a cdga, we need to add a differential δ, which we can choose freely on some (homogeneous) basis. (One example is $\delta = 0$.)

Example 2.1.9. Let $M_\bullet$ be the graded k-module by $M_0 = k\{X\}$ and $M_1 = k\{Y, Z\}$. Then the free graded-commutative algebra $k[M_\bullet]$ is given by

$$k[M_\bullet] = \underbrace{k[X]}_{\deg 0} \oplus \underbrace{k[X] \cdot \{Y, Z\}}_{\deg 1} \oplus \underbrace{k[X] \cdot \{YZ\}}_{\deg 2}.$$

To turn this into a cdga, we need to specify values of $\delta(X), \delta(Y), \delta(Z)$. Since X lives in degree 0, $\delta(X) = 0$. On the other hand Y, Z live in degree 1 so $\delta(Y), \delta(Z) \in k[M_\bullet]_0 = k[X]$, and we have complete freedom to choose such elements. The properties of the differential and multiplication then determine everything else.

In fact, $H_0(k[M_\bullet]) = k[X]/(\delta(Y), \delta(Z))$.

Remark 2.1.10. It's fairly straightforward to adjust these constructions to differential or analytic geometry. In differential geometry, A_0 should be a $\mathcal{C}^\infty$-ring, and the analogue is known as synthetic differential geometry. In analytic geometry, A_0 should be a ring with entire functional calculus (EFC-ring). Pretty much most of the remaining constructions have analogues in such fields, and sometimes are even easier, but we won't reference these similarities (or differences) again. See [EP21] for comments (this part of the notes is based on it anyway).

Now that we've defined the objects and morphisms (and they appear very ordinary), we can go ahead and construct the analogue of the category of k-algebras.

Definition 2.1.11. We denote by $\mathrm{dg}_+\mathrm{Alg}_k$ the **category of cdga which are concentrated in nonnegative degree**.

The opposite category $(\mathrm{dg}_+\mathrm{Alg}_k)^{\mathrm{op}}$ is the **category of affine dg-schemes**, and is denoted by $\mathrm{DG}_+\mathrm{Aff}_k$. We denote elements in the opposite category by $\operatorname{Spec} A_\bullet$ (this is purely formal at the moment, there is no explicit construction with prime ideals, etc... yet).

Remark 2.1.12. Although we have not discussed the geometric picture yet, one should visualize the points of $\operatorname{Spec} A_\bullet$ as just the points of $\operatorname{Spec} H_0(A_\bullet)$, which is an ordinary affine scheme. The rest of the structure coming from the higher cohomologies is in some sense infinitesimal.

2.2 dg-schemes

The next step is to glue affine dg-schemes to form dg-schemes. We might start with the standard: gluing affine dg-schemes along open subschemes, gluing the corresponding modules, etc. But upon further reflection we actually don't really need to, because all of the work is already done for us in standard algebraic geometry – this is exactly what gluing affine schemes and sheaves together is. So we'll just piggyback off of their hard work and define a dg-scheme to just be a chain complex of glued modules over affine schemes – in other words, a chain complex of quasicoherent sheaves.

Definition 2.2.1. A **dg-scheme** consists of a scheme X^0 along with quasicoherent sheaves $\mathcal{O}_X := \{\mathcal{O}_{X,i}\}_{i\geq 0}$ on X^0, satisfying the following conditions:

- $\mathcal{O}_{X,0} = \mathcal{O}_{X^0}$, i.e. the zeroth sheaf is the structure sheaf of the scheme X^0,
- the quasicoherent sheaves are equipped with a cdga structure, consisting of a differential map $\delta : \mathcal{O}_{X,i} \to \mathcal{O}_{X,i-1}$ and a compatible multiplication $- \cdot - : \mathcal{O}_{X,i} \otimes \mathcal{O}_{X,j} \to \mathcal{O}_{X,i+j}$ satisfying the usual conditions.

However, recall that for an affine dg-scheme, the "spectrum" was actually Spec of $H_0(A_\bullet) = \ker(A_1 \xrightarrow{\delta} A_0)$, not all of A_0. So actually the underlying "scheme" is not all of X^0 but the subscheme defined by the ideal $\delta(\mathcal{O}_{X,1}) \subset \mathcal{O}_{X,0}$.

Definition 2.2.2. Define the **underived truncation** $\pi^0 X := \underline{\operatorname{Spec}}_{X^0} H^0(\mathcal{O}_X) \subset X^0$ to be the closed subscheme of X^0 defined by the ideal $\delta(\mathcal{O}_{X,1}) \subset \mathcal{O}_{X,0}$. The underived truncation $\pi^0 X$ is also known as the **classical locus** of X.

Definition 2.2.3. A morphism of dg-schemes is called a **quasi-isomorphism** if $\pi^0 f : \pi^0 X \to \pi^0 Y$ is an isomorphism of schemes inducing the isomorphisms $\mathcal{H}_\bullet(\mathcal{O}_Y) \xrightarrow{\sim} \mathcal{H}_\bullet(\mathcal{O}_X)$ of quasicoherent sheaves on $\pi^0 X = \pi^0 Y$ (these are the homology objects taken in the category of sheaves).

Now the problem comes out with $X^0 \neq \pi^0 X$. The scheme that we actually care about is not all of X^0, so actually in our definition of a dg-scheme, we could have replaced X^0 with an open subset containing $\pi^0 X$; this would give us a quasi-isomorphic dg-scheme. So the "rest" of X^0 is sort of meaningless ambient space which is unwieldly and can get in the way when we try to glue schemes together (since it's just sitting there undefined, can be arbitrarily large, and providing no structure at all). This ends up being too restrictive. The solution is to use derived schemes.

Definition 2.2.4. A **derived (k-)scheme** X is a scheme $\pi^0 X$ and a presheaf $\mathcal{O}_X$ on the site of affine open subschemes of $\pi^0 X$, taking values in $\mathrm{dg}_+\mathrm{Alg}_k$, such that in degree zero we have $H_0(\mathcal{O}_X) = \mathcal{O}_{\pi^0 X}$, and the $H_i(\mathcal{O}_X)$ are quasicoherent $\mathcal{O}_{\pi^0 X}$-modules for all i.

Let me explain it more concretely. First, we are dropping the ambient scheme X^0 in favor of just the underived truncation $\pi^0 X$, which handles the issue from before. Usually a k-scheme would consist of algebras living over each affine open subscheme, satisfying certain compatibility/gluing. Now we have a dga living over each affine open subscheme, again satisfying certain compatibility/gluing, except that now only the zeroth part $H_0(\mathcal{O}_X)$ actually needs to identify with the structure sheaf of our underlying scheme $\pi^0 X$, and the higher order terms do not. What the higher order terms $H_i(\mathcal{O}_X)$ *do* need to do, is to satisfy quasicoherent compatibility with the structure sheaf $\mathcal{O}_{\pi^0 X} = H_0(\mathcal{O}_X)$: namely, for each inclusion $U \hookrightarrow V$ of affine open subschemes of $\pi^0 X$ (this is the data of a presheaf on the site of affine open subschemes of $\pi^0 X$), we need isomorphisms of presheaves of homology groups

$$\mathcal{O}_{\pi^0 X}(U) \otimes_{\mathcal{O}_{\pi^0 X}(V)} H_i(\mathcal{O}_X(V)) \xrightarrow{\sim} H_i(\mathcal{O}_X(U)),$$

or alternatively

$$H_0(\mathcal{O}_X(U)) \otimes_{H_0(\mathcal{O}_X(V))} H_i(\mathcal{O}_X(V)) \xrightarrow{\sim} H_i(\mathcal{O}_X(U)).$$

Note that $\mathcal{O}_X$ refers to the presheaf taking values in $\mathrm{dg}_+\mathrm{Alg}_k$, **not** the structure sheaf of X (as it is commonly used), for an important reason: there is no scheme X here!

Still, dg-schemes and derived schemes are not that far from each other.

Construction 2.2.5. From a dg-scheme $(X^0, \mathcal{O}_X)$, we get a derived scheme $(\pi^0 X, i^{-1}\mathcal{O}_X)$ where $i : \pi^0 X \hookrightarrow X$ is the canonical embedding. So **any dg-scheme will give us a derived scheme** essentially by "only remembering the classical locus."

On the other hand, from a derived scheme $(\pi^0 X, \mathcal{O}_X)$, we get an affine dg-scheme $\operatorname{Spec} \mathcal{O}_X(U)$ for any affine subscheme $U \subset \pi^0 X$ whose underlying schemes are $\operatorname{Spec}(\mathcal{O}_X(U))_0 \supseteq U$. Unfortunately, these carry the "extraneous space" discussed before, and this lingering ambience actually generally prevents us from being able to glue them together into a globalized scheme $X^0 \supseteq \pi^0 X$.

In simpler terms: imagine a dg-scheme is an apple and the fruit inside (without the skin) is the underived truncation. Then by peeling the skin, we are left with only the juicy goodness of the fruit itself, i.e. a derived scheme. But when presented with peeled apple slices, we may never know what the original skin looked like. So we can't "unscramble the egg" by turning a derived scheme back into a dg-scheme.

Remark 2.2.6. There are characterizations using sheaves instead of presheaves, and can be obtained from the data in our definition $(\pi^0 X, \mathcal{O}_X)$ by sheafifying each presheaf $\mathcal{O}_{X,n}$ individually. But this messes up some hypersheaf property, so the quasi-inverse functor is not just the forgetful functor of forgetting the sheaf property, which is why we don't use sheaves directly here.

Remark 2.2.7. According to [EP21], derived schemes as defined here are equivalent to derived Artin or Deligne-Mumford ∞-stacks whose underlying derived stacks are schemes, as in [TV08]. I don't know what those are, but if you do, then this remark might be useful to you.

2.3 Quasicoherent complexes

We've discussed dg-algebras, but we've put off dg-modules for a while. The reason is because we actually care about quasicoherent sheaves on derived schemes, rather than dg-schemes. But now that we have derived schemes, let's review dg-modules.

Definition 2.3.1. Let $A_\bullet \in \mathrm{dg}_+\mathrm{Alg}_k$. An **$A_\bullet$-module (in chain complexes)** is a chain complex $M_\bullet$ of k-modules with a corresponding action of $A_\bullet$.

Explicitly, for all i, j we have a k-bilinear map $A_i \times M_j \to M_{i+j}$ satisfying the usual properties of multiplication, and additionally the chain map condition $\delta_M(am) = \delta_a(a)m + (-1)^{\deg a} a\delta_M(m)$.

This is succinctly summarized by giving a map of the total complex $A_\bullet \otimes_k M_\bullet \to M_\bullet$, compatible with the multiplication on $A_\bullet$.

We denote the category of $A_\bullet$-modules concentrated in nonnegative degree by $\mathrm{dg}_+\mathrm{Mod}_{A_\bullet}$. We have the usual definition of morphisms of $A_\bullet$-modules and quasi-isomorphism.

Now we globalize this.

Definition 2.3.2. Let $(\pi^0 X, \mathcal{O}_X)$ be a derived scheme. The analogue of the previous definition is to look at $\mathcal{O}_X$-modules $\mathcal{F}$ in complexes of presheaves. We say they are **quasicoherent complexes** (sometimes called **homotopy Cartesian modules**) if the homology presheaves $H_i(\mathcal{F})$ are all quasicoherent $\mathcal{O}_{\pi^0 X}$-modules.

Once again, the condition that all $H_i(\mathcal{F})$ are quasicoherent $\mathcal{O}_{\pi^0 X}$-modules is just saying that for every inclusion $U \hookrightarrow V$ of affine open subschemes of $\pi^0 X$, we have isomorphisms

$$H_0(\mathcal{O}_X(U)) \otimes_{H_0(\mathcal{O}_X(V))} H_i(\mathcal{F}(V)) \xrightarrow{\sim} H_i(\mathcal{F}(U)).$$

Note that this may be rephrased in the language of derived functors as saying that we have quasi-isomorphisms

$$\mathcal{O}_X(U) \otimes^L_{\mathcal{O}_X(V)} \mathcal{F}(V) \xrightarrow{\sim} \mathcal{F}(U).$$

2.4 Missing pieces

The intuition from derived categories tells us that we should think of derived schemes X, Y as equivalent if they can be connected by a zigzag of roofs of quasi-isomorphisms:

$$\begin{tikzcd}
 & \bullet \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & & \bullet \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & & \bullet \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & & \bullet \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\
X & & \bullet & & \bullet & & \bullet & & Y
\end{tikzcd} .$$

But that raises the question: how should we define morphisms to be compatible with this notion of equivalence? How should we define gluing? There are many issues that arise, including a glaring one which is that forcibly inverting quasi-isomorphisms gives a "homotopy category" which doesn't have limits and colimits and doesn't behave well with gluing, even in the affine case. If we take I to be the poset of open affine subschemes (as in the definition of a derived scheme), then we could try to form $\mathrm{dg}_+\mathrm{Alg}_k^I$ the category of I-shaped diagrams in $\mathrm{dg}_+\mathrm{Alg}_k$. Then we can invert quasi-isomorphisms to obtain the homotopy category $\mathrm{Ho}(\mathrm{dg}_+\mathrm{Alg}_k^I)$ of $\mathrm{dg}_+\mathrm{Alg}_k^I$, but unfortunately the natural functor $\mathrm{Ho}(\mathrm{dg}_+\mathrm{Alg}_k^I) \to \mathrm{Ho}(\mathrm{dg}_+\mathrm{Alg}_k)^I$ is not an equivalence (in fact it fails for everything but discrete diagrams, i.e. when I is just a set). The tl;dr is that **lots of problems arise when trying to do the standard constructions (i.e. homotopy category, derived category) to define the right category for derived schemes**. Actually the answer is to use ∞-categories, specifically $(\infty, 1)$-categories.

3 Infinity categories, model categories, and consequences for dg-algebras

I don't really want to get into the formality of infinity categories and model categories, so I'll try to minimize the abstractness as much as possible and only bring up the necessary conventions so we can look at some consequences for dg-algebras.

3.1 Pretending to understand infinity categories

There are many equivalent notions of ∞-categories, and as such, there are many ways to get an extremely superficial understanding of them. Let's see a few for intuition.

(1) Perhaps the easiest notion is the concept of a **topological category**, which is a category enriched in topological spaces. Concretely, this just means that we should equip the Hom-sets of a category with the structure of a topological space, so that we have additional information (namely topological information) when dealing with morphisms. Naturally everything should now be phrased in terms of continuity.

The **homotopy category** $\mathrm{Ho}(\mathcal{C})$ of a topological category $\mathcal{C}$ is just the category where we only remember the path components of morphisms, i.e. $\mathrm{Obj}(\mathrm{Ho}(\mathcal{C})) = \mathrm{Obj}(\mathcal{C})$, but $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X, Y) = \pi_0 \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

A functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ of topological categories (of course, assumed to respect topological structure) is a **quasi-equivalence** if F induces isomorphisms on all homotopy groups $\pi_n(\mathrm{Hom}_{\mathcal{C}}(X, Y)) \xrightarrow{\sim} \pi_n(\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(X), \mathcal{F}(Y))$ for all X, Y, as well as an equivalence of homotopy categories (this basically means it's essentially surjective and also induces bijections on π_0 of the Hom-spaces). The upshot is that for functors to be "equivalences" they only need to be weak equivalences on Hom-sets (or rather Hom-spaces), and not true homeomorphisms.

(2) Topological spaces carry lots of information which we don't necessarily always want. A much more effort presentation is given by **simplicial categories**, which have a simplicial set of morphisms between objects. They're very similar to topological categories due to the relationship between simplicial sets and topological spaces, but they are much easier to handle.

(3) The easiest notion to construct is called a **relative category**. These are just pairs $(\mathcal{C}, \mathcal{W})$ where $\mathcal{W} \subset \mathcal{C}$ is a subcategory encoding "some notion of equivalence" which is not as strong as isomorphism. Two very common notions of $\mathcal{W}$ are given by weak equivalences of topological spaces, and quasi-isomorphisms of chain complexes.

The homotopy category $\mathrm{Ho}(\mathcal{C})$ is just the localization of $\mathcal{C}$ at $\mathcal{W}$ (note that this is different from the classical notion of homotopy category in homological algebra, in which only "strong homotopy equivalences" are inverted). The simplicial category is closely related. The drawback of using relative categories is that quasi-equivalences are hard to describe.

One point that [EP21] makes is that when given an infinity category, you can think of it as a topological or simplicial category (since they're easier to visualize), while if you need to construct one, you can just give a relative category (since they're easier to construct formally but harder to understand concretely).

3.2 Model categories

Model categories are pretty much just relative categories with extra structure, and this extra structure helps us compute stuff - namely derived functors. The first and foremost point of relative categories is to specify a class of "weak equivalences" which will be more general than isomorphisms, but the correct notion for our situation. The rest of the structure exists to make the category more concrete and accessible.

Definition 3.2.1. A **model category** is a relative category $(\mathcal{C}, \mathcal{W})$ together with two classes of morphisms called **fibrations** and **cofibrations**, which satisfy certain axioms (inspired by algebraic topology).

A fibration which is also lies in $\mathcal{W}$ (i.e., a weak equivalence) is called a **trivial fibration**, and **trivial cofibrations** are defined in the same way.

We'll skip the exact definitions of fibrations and cofibrations, as well as their properties, and get straight to examples.

Example 3.2.2. Suppose $\mathcal{C}$ is a category with limits and colimits. Then the **trivial model structure** is just the model structure where weak equivalences are isomorphisms, and fibrations and cofibrations are just all morphisms.

Example 3.2.3 ($\mathrm{dg}_+\mathrm{Alg}_k$)**.** One important model structure on $\mathrm{dg}_+\mathrm{Alg}_k$ is as follows.

- Weak equivalences are qquasi-isomorphisms.
- Fibrations are maps which are surjective in strictly positive degree, i.e. maps $A_\bullet \to B_\bullet$ such that $A_i \twoheadrightarrow B_i$ for all $i > 0$.
- Cofibrations are maps $A_\bullet \to B_\bullet$ which have the left lifting property with respect to trivial fibrations, i.e. for any trivial fibration $X_\bullet \to Y_\bullet$ then for any commutative square

  $$\begin{tikzcd} A_\bullet \arrow[r] \arrow[d, "f"'] & X_\bullet \arrow[d] \\ B_\bullet \arrow[r] \arrow[ur, dashed, "\exists"] & Y_\bullet \end{tikzcd}$$

  there should exist a map $B_\bullet \to A_\bullet$ still making the diagram commute.

  A more concrete way to understand cofibrations are as retracts of quasi-free maps. A quasi-free map is a map $A_\bullet \to B_\bullet$ such that $B_\bullet$ is freely generated as a graded-commutative $A_\bullet$-algebra.

Example 3.2.4 ($\mathrm{DG}_+\mathrm{Aff}_k$)**.** Since $\mathrm{DG}_+\mathrm{Aff}_k = (\mathrm{dg}_+\mathrm{Alg}_k)^{\mathrm{op}}$, we give it the opposite model structure, which swaps fibrations and cofibrations.

Example 3.2.5 (Complexes of R-modules in nonnegative degree)**.** Consider the category of nonnegatively graded chain complexes of R-modules. The **projective model structure** is given as follows.

- The weak equivalences are quasi-isomorphisms.
- The fibrations are chain maps which are surjective in strictly positive degree.
- The cofibrations are maps $M_\bullet \hookrightarrow N_\bullet$ such that $N_\bullet/M_\bullet$ is a complex of projective R-modules.

The homotopy category is the full subcategory of $D(R-\mathsf{mod})$ of nonnegatively graded chain complexes.

Example 3.2.6 (Complexes of R-modules in nonpositive degree)**.** There is also an **injective model structure** similar to the previous example.

- The weak equivalences are quasi-isomorphisms.
- The cofibrations are chain maps which are injective in strictly negative degree.
- The fibrations are surjective maps with levelwise inject kernel.

The homotopy category is the full subcategory of $D(R-\mathsf{mod})$ of nonpositively graded chain complexes.

Example 3.2.7 (All complexes of R-modules)**.** The classical picture is to **construct the derived category $D(R)$ by inverting all quasi-isomorphisms amongst all chain complexes**. The previous two examples handle the cases where the chain complexes are concentrated entirely in nonnegative or nonpositive degree. To generalize to all complexes, we just need to put more restrictions on either the cofibrations or fibrations, but the end result is that the homotopy category is indeed $D(R\text{--}\mathsf{mod})$.

The upshot of defining these model structures is that we're able to compute stuff using the notions of fibrations and cofibrations.

Definition 3.2.8. Let $\mathcal{C}$ be a model category. An object $X \in \mathcal{C}$ is **fibrant** if the map to the final object $X \to f$ is a fibration; it is **cofibrant** if the map from the initial object $i \to X$ is a cofibration.

Oftentimes we want to work with fibrant and cofibrant objects rather than arbitrary objects. Therefore we need to replace some object $X \in \mathcal{C}$ with an equivalent (co)fibrant object. A **fibrant replacement** of A is some fibrant object $\widehat{A}$ with a weak equivalence $A \to \widehat{A}$. A **cofibrant replacement** of A is some cofibrant object $\widetilde{A}$ with a weak equivalence $\widetilde{A} \to A$.

Example 3.2.9. In Example 3.2.3 we gave a model structure on $\mathrm{dg}_+\mathrm{Alg}_k$. With this model structure, every object is fibrant.

The **main intuition for (co)fibrant replacements is injective and projective resolutions.**

3.3 Derived functors in model categories

One important use of model structures is to give conditions for the existence of **derived functors** and methods for computing them.

Definition 3.3.1. A functor $G : \mathcal{C} \to \mathcal{D}$ of model categories is **right Quillen** if it has a left adjoint F (i.e., it is a right adjoint) and preserves fibrations and trivial cofibrations.

Duually, F is **left Quillen** if it has a right adjoint (i.e., it is a left adjoint) and F preserves cofibrations and trivial fibrations.

In this case, $F \dashv G$ is a **Quillen adjunction**.

The intuition for these comes from **right exact and left exact functors**. In the case of the derived category, left adjoints are right exact and give rise to left derived functors. **In the case of model categories, left Quillen functors are left adjoints and give rise to left-derived functors** and similarly for right Quillen.

Theorem 3.3.2. *If G is right Quillen, then the **right-derived functor** RG exists and sends $A \mapsto G(\widehat{A})$ where $\widehat{A}$ is a fibrant replacement.*

*If F is left Quillen, then the **left-derived functor** LG exists and sends $A \mapsto F(\widetilde{A})$ where $\widetilde{A} \to A$ is a cofibrant replacement.*

Remark 3.3.3. If you don't like the arbitrary choice of a fibrant replacement, we can even take fibrant replacements functorially, but since right Quillen functors preserve weak equivalences between fibrant objects, it turns out this isn't strictly necessary (on objects at least).

Example 3.3.4. The global sections functor Γ is left-exact and is a right-adjoint to Spec. The standard way to define $R\Gamma$ (which computes sheaf cohomology) is to take an injective resolution. But actually fibrant replacement in the model category of nonnegatively graded cochain complexes is the same thing as taking an injective resolution - so it turns out that the classical procedure exactly matches our procedure of taking a fibrant replacement to the right Quillen functor Γ!

Definition 3.3.5. The **homotopy limit** $R\varprojlim_I$ is the right-derived functor of the limit functor $\varprojlim_I : \mathcal{C}^I \to \mathcal{C}$.

For diagrams $X \to Y \leftarrow Z$, we denote the **homotopy fiber product** by $X \times_Y^h Z$.

Lemma 3.3.6. *If Y is fibrant, we can compute the homotopy fiber product $X \times_Y^h Z$ by taking fibrant replacements of X and Z.*

This is dual to the notion of taking a derived tensor product by replacing both components by projective resolutions in a (classical) derived category.

3.4 Some consequences for dg-algebras

Now that we are armed with quite a lot of abstract nonsense, let's see what it can do when applied to dg-algebras.

The first step is noting the embedding $\mathrm{Alg}_k \hookrightarrow \mathrm{dg}_+\mathrm{Alg}_k$ by embedding each algebra as a chain complex concentrated in degree 0. We also have a map $\mathrm{dg}_+\mathrm{Alg}_k \to \mathrm{Ho}(\mathrm{dg}_+\mathrm{Alg}_k)$ by passing to the homotopy category.

Proposition 3.4.1. *The composition $\mathrm{Alg}_k \hookrightarrow \mathrm{dg}_+\mathrm{Alg}_k \to \mathrm{Ho}(\mathrm{dg}_+\mathrm{Alg}_k)$ is fully faithful.*

In other words, for an affine scheme X and a derived affine scheme Y, we have

$$\mathrm{Hom}_{\mathrm{Ho}(\mathrm{DG}_+\mathrm{Aff}_k)}(X, Y) \cong \mathrm{Hom}_{\mathrm{Aff}}(X, \pi^0 Y).$$

Another very important consequence of the model structure on $\mathrm{dg}_+\mathrm{Alg}_k$ is how derived functors work. In the next subsection we'll see how derived tensor products work.

3.5 Derived tensor products

If $A_\bullet, B_\bullet \in \mathrm{dg}_+\mathrm{Alg}_k$ then we can treat them simply as complexes, forgetting the algebra structure on them. There is already the notion of the tensor product of complexes, which is the complex $(A_\bullet \otimes_k B_\bullet)_\bullet$ whose nth graded component is $\sum_{i+j=n} A_i \otimes_k B_j$, and it is not difficult to specify the differential. It turns out there is no problem defining a multiplication structure either: just $(a \otimes b) \cdot (a' \otimes b') = (-1)^{(\deg a')(\deg b)}(aa' \otimes bb')$. Therefore we have a notion of tensor product of dgas, which agrees with the tensor product of complexes when we forget the algebra structure.

Our goal is to define the **derived tensor product**, in much the same way as we define it in the derived category (which is by taking projective resolutions of both factors and then applying the usual tensor product of complexes to these resolutions).

The first step in defining a derived functor is to check whether it is right Quillen or left Quillen.

Lemma 3.5.1. *The functor* $\otimes_k : \mathrm{dg}_+\mathrm{Alg}_k \times \mathrm{dg}_+\mathrm{Alg}_k \to \mathrm{dg}_+\mathrm{Alg}_k$ *is left Quillen. It has a right adjoint* $A \mapsto (A, A)$*.*

Now the general theory of §3.3 (specifically, Theorem 3.3.2) implies that the left derived functor exists, and gives an explicit description of it.

Definition 3.5.2. The **derived graded tensor product** $\otimes_k^L : \mathrm{Ho}(\mathrm{dg}_+\mathrm{Alg}_k) \times \mathrm{Ho}(\mathrm{dg}_+\mathrm{Alg}_k) \to \mathrm{Ho}(\mathrm{dg}_+\mathrm{Alg}_k)$ is defined to be the left derived functor of $\otimes_k$.

Explicitly, to compute $A_\bullet \otimes_k^L B_\bullet$, take cofibrant replacements $\widetilde{A}_\bullet$ and $\widetilde{B}_\bullet$ and then take the usual tensor product:

$$A_\bullet \otimes_k^L B_\bullet = \widetilde{A}_\bullet \otimes_k \widetilde{B}_\bullet.$$

Remark 3.5.3. Recall that k here can be any $\mathbb{Q}$-algebra, not just a field, so this construction is not as obvious as it might seem.

Actually, when we take derived tensor products in a derived category of coherent sheaves, we don't take projective resolutions of the factors (in large part because they usually don't exist): we get away with the much easier procedure of taking a locally free resolution. The same general principle holds here: we don't actually need cofibrant replacements, but rather something much less complicated.

Definition 3.5.4. Let $A_\bullet \in \mathrm{dg}_+\mathrm{Alg}_k$ be a cdga and $M_\bullet$ an $A_\bullet$-module. Then $M_\bullet$ is called **quasi-flat** if the underlying graded module of $M_\bullet$ is flat over the the underlying graded algebra of $A_\bullet$.

Proposition 3.5.5. *To compute* $A_\bullet \otimes_k^L B_\bullet$*, we can take a quasi-flat replacement (over* k*) of either of the two factors, then apply the usual tensor product* $\otimes_k$*.*

This holds more generally over an arbitrary base $R_\bullet \in \mathrm{dg}_+\mathrm{Alg}_k$, not just an honest ring k:

Proposition 3.5.6. *To compute* $A_\bullet \otimes_{R_\bullet}^L B_\bullet$ *we can take a quasi-flat replacement of either of the two factors, then apply the usual tensor product* $\otimes_{R_\bullet}$*.*

Now the tensor product in the category of k-algebras plays the dual role to the fiber product in the category of k-schemes. This is still true in $\mathrm{dg}_+\mathrm{Alg}_k$:

Definition 3.5.7. In the opposite category $\mathrm{DG}_+\mathrm{Aff}_k$, we denote these derived tensor products as **homotopy pullbacks**, and we write $X \times_Z^h Y := \mathrm{Spec}(A_\bullet \otimes_{C_\bullet}^L B_\bullet)$ where $X = \mathrm{Spec}\, A_\bullet$, $Y = \mathrm{Spec}\, B_\bullet$, and $Z = \mathrm{Spec}\, C_\bullet$.

Example 3.5.8. This example will work out the details of

$$\{0\} \times^h_{\mathbb{A}^1} \{0\}.$$

In the classical setting, this is the fiber product of the two maps $\{0\} \hookrightarrow \mathbb{A}^1 \hookleftarrow \{0\}$, which is clearly just $\{0\} \cong \operatorname{Spec} k$ (alternatively, this is easily computed as $\operatorname{Spec} k \otimes_{k[t]} k \cong \operatorname{Spec} k$). But in the derived setting, this is quite interesting!

First we need a quasi-flat resolution of k over $k[t]$. This is given by the cdga $k[t,s]$ where $\deg t = 0$ and $\deg s = 1$ and $\delta(s) = 1$, so that the complex is $k[t] \cdot s \to k[t]$, and $s \mapsto 1$. (In fact, this is even cofibrant.) Now we can apply the usual tensor product of dgas:

$$k \otimes^L_{k[t]} k = k[t,s] \otimes_{k[t]} k = k[s] = (k \cdot s \xrightarrow{s \mapsto 1} k).$$

Note that applying the 0th homology functor H_0 to the complex recovers k, the underived fiber product (=tensor product of rings). What the **derived** tensor product is telling us here is that there are **virtual points** in the derived scheme $\operatorname{Spec} k[s]$ which may have positive or negative weights depending on which graded component they show up in. Applying the Euler characteristic to the complex $k \cdot s \xrightarrow{s \mapsto 1} k$, we obtain $1 - 1 = 0$ (as it's just $k \to k$ as a complex of vector spaces), so the derived intersection "should" contain "zero total points." But that means that there is the negatively weighted "virtual point" counteracting our "legitimate" point given in degree 0 that we can recover by applying H_0!

We can also talk about the "virtual dimension," given informally by the Euler characteristic of the generators. Here, $k[s]$ has a single generator in degree 1 (which is odd), so the virtual dimension is -1. This agree with the notion of "intersecting two codimension 1 subschemes of a dimension 1 scheme," which realistically "should" be dimension $1 - 1 - 1 = -1$, but in the underived setting it's dimension 0 (and negative dimension doesn't make sense anyway).

Remark 3.5.9. These properties are instances of a more general phenomenon of generalizing the properties from the classical world to the derived world.

Example 3.5.10. More generally, let's consider the derived intersection

$$\{a\} \times^h_{\mathbb{A}^1} \{0\} = \operatorname{Spec}(k \otimes_{a \leftarrow\!\shortmid t, k[t], t \mapsto 0} k).$$

In the underived setting, this is just $\operatorname{Spec} 0 = \emptyset$ whenever $a \neq 0$, and $\operatorname{Spec} k = \{0\}$ when $a = 0$. It's kind of weird that this has a different size as a varies. Indeed, this is exactly a strange issue that comes up in intersection theory: when we multiply two divisors, we need to swap out one of the divisors for a linearly equivalent divisor which "intersects transversely" in order to count it geometrically, which always seemed pretty unsatisfactory to me.

Now let's consider the derived setting. We have the quasi-flat (even cofibrant) $k[t]$-algebra $k[s,t]$ where $\deg t = 0$, $\deg s = 1$, and $\delta(s) = t - a$. This is quasi-isomorphic to k as a $k[t]$-module corresponding to

$k = k[t]/(t-a)$. Now we compute the derived intersection as

$$k \otimes_{a \leftarrow t, k[t], t \mapsto 0} k = k[t,s] \otimes_{t \leftarrow t, k[t], t \mapsto 0} k = k[s],$$

where $\delta(s) = t - a$ in all instances where t exists and $\delta(s) = -a$ where t doesn't exist.

Now, when $a = 0$, then we recover the example from above. However, when $a \neq 0$, then $k[s]$ is quasi-isomorphic to the 0 complex, hence the derived intersection is quasi-isomorphic to $\operatorname{Spec} 0 = \emptyset$. This obvious has no points. But as we discussed in the example with $a = 0$, the total number of points in the derived intersection is still 0! This corresponds to our intuition of intersecting two generic points in $\mathbb{A}^1$. Consequently, it categorifies Serre's intersection numbers, giving a more fulfilling answer to why intersection numbers behave the way they do.

Our last example is the important example of a **derived loop space**.

Definition 3.5.11. Let $X \in \mathrm{DG}_+\mathrm{Aff}_k$. The **derived loop space** of X is

$$\mathcal{L}X := X \times^h_{X \times X} X,$$

where the two maps are the diagonal embeddings $\Delta : X \hookrightarrow X \times X$.

Example 3.5.12. Let's compute $\mathcal{L}\mathbb{A}^1 = \mathbb{A}^1 \times^h_{\mathbb{A}^1 \times \mathbb{A}^1} \mathbb{A}^1$. This is the Spec of

$$(k[x,y]/(x-y) \otimes^L_{k[x,y]} (k[x,y]/(x-y)).$$

We need a quasi-flat replacement: we can take $k[x,y,s]$ with $\deg x = \deg y = 0$, and $\deg s = 1$, with $\delta(s) = x - y$ (this is even cofibrant!). This gives us the complex $k[x,y] \cdot s \to k[x,y]$. Now we apply the usual tensor product to find that

$$\mathcal{L}\mathbb{A}^1 = \mathbb{A}^1 \times^h_{\mathbb{A}^1 \times \mathbb{A}^1} \mathbb{A}^1 = \operatorname{Spec}(k[x,y]/(x-y) \otimes^L_{k[x,y]} k[x,y,s]) = \operatorname{Spec} k[x,s],$$

where $k[x,s]$ is the cdga with $\deg x = 0$, $\deg s = 1$, and $\delta(s) = x - x = 0$. Therefore this is the cdga $k[x] \cdot s \xrightarrow{0} k[x]$, with underlying scheme $\operatorname{Spec} H_0(k[x,s]) = \operatorname{Spec} k[x]$ but "virtual dimension" $1 - 1 = 0$ (which is expected, given that X has codim $\frac{\dim X \times X}{2}$ in $X \times X$, so intersecting it with another codim-$\frac{\dim X \times X}{2}$-scheme should give us a dimension zero scheme).

Example 3.5.13. Let X be an arbitrary smooth affine scheme. Then

$$\mathcal{L}X = \operatorname{Spec}(\Omega_X^{\dim X} \xrightarrow{0} \cdots \xrightarrow{0} \Omega_X^2 \xrightarrow{0} \Omega_X^1 \xrightarrow{0} \mathcal{O}_X).$$

This is a strengthening of the HKR isomorphism!

3.6 Postnikov towers

The main purpose of this subsection is to justify the idea that the geometric part of a cdga is mostly just the scheme $\operatorname{Spec} H_0(-)$, with the "rest of the complex" just being "infinitesimal." The basic idea is to write a cdga $A_\bullet$ as limit of a sequence of homotopy square-zero thickenings.

Definition 3.6.1. Fix $A_\bullet \in \mathrm{dg}_+\mathrm{Alg}_k$. Then define the cdga $(P_nA)_\bullet$ by

$$(P_nA)_i = \begin{cases} A_i & i \le n, \\ \mathrm{im}(A_{n+1} \xrightarrow{\delta} A_n) & i = n+1, \\ 0 & i > n+1. \end{cases}$$

These form the diagram

$$\begin{tikzcd}
& A_\bullet \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] \arrow[drrr] \arrow[drrrr] & & & & & \\
\cdots \arrow[r] & (P_4A)_\bullet \arrow[r] & (P_3A)_\bullet \arrow[r] & (P_2A)_\bullet \arrow[r] & (P_1A)_\bullet \arrow[r] & (P_0A)_\bullet \arrow[r] & 0.
\end{tikzcd}$$

The key idea is that if we define

$$(Q_nA)_i = \begin{cases} A_i & i < n, \\ \mathrm{coker}(A_{n+1} \xrightarrow{\delta} A_n) & i = n, \\ 0 & i > n \end{cases}$$

then we obtain the factorization

$$\begin{tikzcd}
(P_nA)_\bullet \arrow[rr] \arrow[dr] & & (P_{n-1}A)_\bullet \\
& (Q_nA)_\bullet \arrow[ur] &
\end{tikzcd}$$

where $(P_nA)_\bullet \to (Q_nA)_\bullet$ is a trivial fibration and $(Q_nA)_\bullet \to (P_{n-1}A)_\bullet$ is a square-zero extension with kernel $H_n(A_\bullet)[-n]$. This means that we can view $\mathrm{Spec}\, A_\bullet$ as a formal infinitesimal thickening of $\mathrm{Spec}\, H_0(A_\bullet)$, as the rest of it is just a bunch of square-zero extensions and trivial fibrations.

If we assume some finiteness conditions, then this viewpoint strengthens to a true formal completion.

Proposition 3.6.2. *Suppose $A_\bullet \in \mathrm{dg}_+\mathrm{Alg}_k$ is such that A_0 is Noetherian and each A_n is a finitely generated A_0-module. Let $I = \ker(A_0 \to H_0(A_\bullet)) \simeq \mathrm{coker}(A_1 \to A_0)$. Then the natural map*

$$A_\bullet \to \varprojlim_n A_\bullet / I^n A_\bullet$$

is a quasi-isomorphism.

4 Differential graded Lie algebras and deformation theory

The main point of this section is that broadly speaking, **deformations (say, of a scheme) are controlled by differential graded Lie algebras**. That's a pretty big statement. Naturally, we will investigate what differential graded Lie algebras are and how they control deformation theory. This mainly follows [ABCW15, §4].

While reading this section, I would strongly emphasize that you should keep the following main picture in mind, since this is a fairly technical subject and it's easy to get lost in all of the machinery. Supposedly, deformation theory (briefly discussed in §4.1) is controlled by objects called dgLas. In order to study them, we need to "categorify" the properties of deformation theory, leading to the definition of a deformation functor in Definition 4.2.1. The key point is that a deformation functor is just an abstraction of the functor which literally gives deformations of a scheme, discussed in Example 4.2.2. In §4.4 I finally tell you what a dgLa is, but you still don't know how this relates to deformation theory. The main example to understand is Example 4.5.1, which should give you an idea for why we worked so hard to define all sorts of abstract stuff in §4.4. Finally, the remainder of §4.5 explains how dgLas are related to deformation functors, and the end of the subsection gives some examples which highlight how you can use dgLas to study classical examples of deformation theory.

4.1 Basic deformation theory

Let k be an algebraically closed field of characteristic 0. In deformation theory, we have some object (say, a scheme) defined over k which we wish to deform to some object defined over some "fuzzy" version of k (the notable example being $k[\varepsilon]/\varepsilon^2$). The correct notion of "fuzzy point" is Artinian local k-algebras with residue field k.

Definition 4.1.1. Let Art_k denote the category of Artinian local k-algebras with residue field k. The morphisms are local k-algebra homomorphisms. For $A \in \mathsf{Art}_k$, let $\mathfrak{m}_A$ denote its unique maximal ideal.

Note the following facts about A:

- A is a finite-dimensional k-vector space.
- $\mathfrak{m}_A$ is nilpotent.
- $A/\mathfrak{m}_A \cong k$ as k-algebras.

The point of working in Art_k is to work with more general rings than just $k[\varepsilon]/\varepsilon^n$.

Now we have a canonical map $A \to A/\mathfrak{m}_A \cong k$, which is a map of k-algebras. This means we have a map $\operatorname{Spec} k \to \operatorname{Spec} A$. A deformation of a k-algebra R should be a flat A-algebra $\widetilde{R}$ such that $\widetilde{R}/\mathfrak{m}_A \cong R$ as k-algebras. In other words, we allow "infinitesimal structure" on $\widetilde{R}$, which is given by the fuzziness of $\operatorname{Spec} A$ compared to the non-fuzziness of $\operatorname{Spec} k$, but if we delete the fuzziness by modding out by $\mathfrak{m}_A$, then we should recover our original R. Globalizing this, a k-scheme X is equipped with a map $X \to \operatorname{Spec} k$, and so a deformation of X (over A) should be some scheme $\widetilde{X}$ living over $\operatorname{Spec} A$ which "remembers its origins from $X \to \operatorname{Spec} k$."

Definition 4.1.2. A **deformation** of a k-scheme X over $\operatorname{Spec} A$ is an A-scheme $\widetilde{X}$, flat over $\operatorname{Spec} A$, which pulls back to the k-scheme X over $\operatorname{Spec} k$ (as a pullback):

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \widetilde{X} \arrow[d] \\
\operatorname{Spec} k \arrow[r] & \operatorname{Spec} A.
\end{tikzcd}$$

It is known that if X is a smooth affine scheme, then any deformation of X over A is trivial. In other words, any deformation of X over A is isomorphic to $X \times_{\operatorname{Spec} k} \operatorname{Spec} A$! However, this is certainly not the case in general, and the theory is governed by obstructions in cohomology groups (e.g. of conormal bundles, etc.); see [Har10].

The naive way to build up deformations over $k[\varepsilon]/\varepsilon^n$ is to consider the deformations of $k[\varepsilon]/\varepsilon^{n-1}$ and extend one level further. Indeed, in general, we want to use this so-called "small extensions" to build up deformations.

Definition 4.1.3. Let $\pi : A \twoheadrightarrow B$ be a surjection in Art_k with kernel J. We say that the exact sequence $0 \to J \to A \to B \to 0$ is a **square zero extension** if $J^2 = 0$. Note that J naturally has the structure of a B-module and also is a k-vector space. Furthermore, we say that a square zero extension is **small** is $\dim_k J = 1$ and $J \cdot \mathfrak{m}_A = 0$.

The point is that we never really need to extend deformations by large degrees. Instead we can basically just find deformations one step at a time using small extensions. This is essentially formalizing the intuition that a deformation over $k[\varepsilon]/\varepsilon^n$ can be built one step at a time, first using a deformation over $k[\varepsilon]/\varepsilon^i$ and then extending it to a deformation $k[\varepsilon]/\varepsilon^{i+1}$ until we reach n.

Proposition 4.1.4. *Every surjection in Art_k facts as a finite composition of small extensions.*

Summary (tl;dr): deformations from a field k are done over some Artinian local k-algebra with residue field k, which is essentially a "fuzzy point" around $\operatorname{Spec} k$; these are generalizations of $k[\varepsilon]/\varepsilon^n$. Then a deformation of a k-scheme X over an Artinian local k-algebra A, is some A-scheme $\widetilde{X}$ which recovers X upon "deleting the fuzz," i.e. modding out by the maximal ideal. To build up a deformation over $k[\varepsilon]/\varepsilon^n$, we might first find a first-order deformation over $k[\varepsilon]/\varepsilon^2$, then extend to a second-order deformation over $k[\varepsilon]/\varepsilon^3$, etc. This intuition carries over to Artinian local k-algebras by Proposition 4.1.4.

4.2 Deformation functors

This subsection aims to set up a general framework for studying deformations. The basic idea is that we should have a functor which controls the deformation theory of say a k-scheme, so that for any $A \in \mathsf{Art}_k$, then $F(A)$ is just the deformations over A.

Informally, a deformation functor is a functor $\mathsf{Art}_k \to \mathsf{Set}$ which mimics all of the crucial properties of a deformation functor. We'd like them to exactly coincide with the functors sending $A \mapsto \{\text{deformations over } A\}$ but we'll have to wait and see how that plays out. For now, all we can do is encode the crucial properties.

Definition 4.2.1. A **deformation functor** is a functor $F : \mathsf{Art}_k \to \mathsf{Set}$ with $F(k)$ being the one element set, and also satisfying the following properties regarding the map η defined as follows. Let $A, B, C \in \mathsf{Art}_k$ with $B \to A$ and $C \to A$ maps in Art_k. We have a natural map $\eta : F(B \times_A C) \to F(B) \times_{F(A)} F(C)$ between F of the pullback, and the pullback of F. Then F must satisfy:

- If $B \twoheadrightarrow A$ is a surjection then η is a surjection.
- If $B = k$ then η is a bijection.

Lastly, F is **homogeneous** if η is a bijection whenever $B \twoheadrightarrow A$ is a surjection.

Proposition 4.1.4 implies that we need only check these properties on small extensions.

Example 4.2.2. Let X be a k-scheme. To no one's surprise, **the canonical example of a deformation functor is the functor**

$$\mathrm{Def}_X : \quad X \mapsto \{\text{deformations of } X \text{ over } A\}/\{\text{isomorphism classes of deformations}\}.$$

After all, if that didn't qualify as a deformation functor, we should probably change the name.

It's not clear what special properties the class of functors $\{\mathrm{Def}_X \mid X \text{ is a } k\text{-scheme}\}$ shares. Our best bet is to try to characterize the most important abstract properties that this class of functors has, which is what the definition of a deformation functor is.

Definition 4.2.3. Let F be a deformation functor. The **tangent space to** F is the set $TF \coloneqq F(k[\varepsilon]/\varepsilon^2)$, and its elements are called **first-order deformations**. Since k is a field, then TF naturally has the structure of a k-vector space (coming from the vector space structure on $k[\varepsilon]/\varepsilon^2$), so a natural transformation $\varphi : F \to G$ induces a linear map $\mathrm{d}\varphi : TF \to TG$, called the **differential** of φ.

The name comes from the following example.

Example 4.2.4. For $R \in \mathsf{Art}_k$ and $F = \mathrm{Hom}(R, -)$ then TR is just the Zariski tangent space of $\operatorname{Spec} R$.

Definition 4.2.5. A deformation functor F is called **smooth** if for every surjection $B \twoheadrightarrow A$ in Art_k (equivalently, for every small extension $B \twoheadrightarrow A$) the map $F(B) \to F(A)$ is surjective.

Definition 4.2.6. An **obstruction theory** for a deformation functor F is the data of a k-vector space V, called the **obstruction space**, and an **obstruction map** $v_\xi : F(A) \to V \otimes I$ for every small extension $\xi = (0 \to I \to B \to A \to 0)$ satisfying the following conditions:

- If $x \in F(A)$ can be lifted to $y \in F(B)$, then $v_\xi(x) = 0$ (**liftings only exist if x has no obstruction** but in general the converse is not true; if this is an iff, then the obstruction theory is called **complete**).

- The obstruction maps v_ξ must commute with morphisms of small extensions. More concretely, a morphism of small extensions ξ_1 and ξ_2 is just the commutative diagram
$$\begin{tikzcd}
\xi_1 : & 0 \arrow[r] & I_1 \arrow[r] \arrow[d, "f_I"] & B_1 \arrow[r] \arrow[d, "f_B"] & A_1 \arrow[r] \arrow[d, "f_A"] & 0 \\
\xi_2 : & 0 \arrow[r] & I_2 \arrow[r] & B_2 \arrow[r] & A_2 \arrow[r] & 0,
\end{tikzcd}$$
such that $v_{\xi_2}(f_A(x)) = (\mathrm{id}_V \otimes f_I)(v_{\xi_1}(x))$ for all $x \in F(A_1)$.

Note that if F is smooth, then all obstruction maps v_ξ are trivial.

Remark 4.2.7. Note that these definitions are actually fairly weak as far as what we'd ideally like an "obstruction theory" to be. Unfortunately, this is the price to pay for working in generality.

This is just the abstract and general version of a classical example in algebraic geometry: the first-order deformations of a scheme being controlled by a map to the cohomology group of the conormal bundle.

However, this is a small issue: if a functor admits a complete obstruction theory V, then we can actually just embed V into any larger space and that would also define a complete obstruction theory. Therefore we just want the minimal complete obstruction theory.

Definition 4.2.8. A morphism of obstruction theories is a linear map $\varphi : V \to W$ such that $w_\xi = (\varphi \otimes \mathrm{id})v_\xi$ for every small extension ξ. An obstruction theory is **universal** if it admits a morphism to every other obstruction theory, formalizing the intuition that we can construct another obstruction theory via $V \hookrightarrow$ any larger space containing V.

The main result:

Theorem 4.2.9. *Every deformation functor admits a universal obstruction theory.*

4.3 Representability

As with any time we're dealing with functors, we want to know if they're **representable**, i.e. isomorphic to $\mathrm{Hom}(X, -)$ for some object X.

Proposition 4.3.1. *Any representable functor* $\mathsf{Art}_k \to \mathsf{Set}$ *is a homogeneous deformation functor. (Unfortunately, not every deformation functor is representable.)*

We have to be slightly more generous if we something like a bijection. We won't have representable functors, but we'll have a slight weakening of it:

Definition 4.3.2. Let $\widehat{\mathsf{Art}_k}$ be the category of complete Noetherian local k-algebras with residue field k, and homomorphisms of local k-algebras. Any functor $F : \mathsf{Art}_k \to \mathsf{Set}$ defines a functor $\widehat{F} : \widehat{\mathsf{Art}_k} \to \mathsf{Set}$ by $\widehat{F}(A) = \varprojlim F(A/\mathfrak{m}_A^n)$. We say that F is **prorepresentable** if $\widehat{F}$ is representable.

Theorem 4.3.3. *A deformation functor F is prorepresentable iff it is homogeneous.*

4.4 Background on Lie algebras and dgLas

This subsection reviews some basic results in Lie theory, so that we can apply dgLas in earnest to deformation theory.

Recall that a **Lie algebra** over k is a vector space V along with a bilinear antisymmetric **Lie bracket** $[,] : L \otimes_k L \to L$ satisfying the Jacobi identity: $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$.

Definition 4.4.1. A **differential graded Lie algebra** (dgLa) is a $\mathbb{Z}$-graded vector space L together with a bilinear Lie bracket $[,] : L^i \otimes L^j \to L^{i+j}$ and a differential $\delta : L^i \to L^{i+1}$ satisfying for all homogeneous elements $x, y, z \in L$:

- $[x,y] = (-1)^{|x|\cdot|y|+1}[y,x]$;
- $(-1)^{|x|\cdot|z|}[x,[y,z]] + (-1)^{|y|\cdot|x|}[y,[z,x]] + (-1)^{|z|\cdot|y|}[z,[x,y]]$ (graded Jacobi identity);
- $\delta[x,y] = [\delta x, y] + (-1)^{|x|}[x, \delta y]$ (graded Leibniz rule).

A **morphism of dgLas** is a morphism of graded vector spaces which commutes with the bracket and the differential. Differential graded Lie algebras over k and their morphisms form a category dgLa_k.

Unsurprisingly, a dgLa is essentially just a combination of a differential graded algebra with a Lie algebra. It is essentially the chain complex version of a Lie algebra in the same way that a dga is the chain complex version of an algebra.

Remark 4.4.2. Note that here, L is *cohomologically graded* since δ raises the degree. One could easily make it homologically graded by setting $L'_i := L_{-i}$.

From here on in, a Lie algebra will mean either an ordinary Lie algebra or a dgLa. (DgLa will only refer to dgla.)

Definition 4.4.3. The **lower central series** of a Lie algebra V is a sequence of Lie subalgebras $V^1 = V$, and $V^i := [V, V^{i-1}]$. A Lie algebra V is **nilpotent** if the lower central series terminates (i.e. becomes all zeroes after some point).

We'll make a passing reference that just as how a (finite-dimensional semisimple) Lie algebra integrates to a simply connected Lie group, so too does a dgLa integrate to a differential graded Lie group (and conversely a differential graded Lie group differentiates to its dgLa): see [JKPS22]. So we can speak of the "Lie group" of a dgLa.

Theorem 4.4.4 (Baker-Campbell-Hausdorff)**.** *Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. For $X, Y \in \mathfrak{g}$, then $\log(\exp(X)\exp(Y))$ can be expressed in terms of commutators of X and Y, give as the following sum:*

$$\log(\exp(X)\exp(Y)) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{2}\left([X,[X,Y]] + [Y,[Y,X]]\right) + \cdots .$$

The point is that **nilpotent Lie algebras are good precisely because we can make sense of the above theorem.**

Proposition 4.4.5. *If V is nilpotent, then the infinite sum in the right hand side of Theorem 4.4.4 is actually finite, hence defines a binary product on V denoted by $*$. This induces the structure of a group on $(V, *)$ which we often denote by $\exp(V)$.*

Another equivalent way to write $\exp(V)$ is as the set $\{e^v \mid vV\}$ endowed with multiplication $e^v \cdot e^w = e^{v*w}$.

So **when V is nilpotent, we can define an alternative group structure** on V (the original group structure being addition, as V is a k-vector space). The motivation for defining this alternative group structure is that **the alternative multiplication $*$ coincides with a natural "adjoint" action of V**, for which we want to be able to compose two "adjoint" actions into another "adjoint" action. Clearly, addition does not work here, so we need this alternative group structure.

Example 4.4.6. Let $V = \mathfrak{u}_n$ be the upper-triangular $n \times n$ matrices. Then V is nilpotent and the map $\exp(V) \to GL_n$ given by $e^X \mapsto \sum_{k \geq 0} \frac{X^k}{k!}$ is a group homomorphism, taking $\mathfrak{u}_n \xrightarrow{\sim} U_n$, the group of strictly upper-triangular matrices (with 1s on the diagonal).

Remark 4.4.7. Here we see why the hypothesis that k has characteristic 0: the exponential wouldn't make sense in characteristic p, as we cannot divide by $p!$.

The 0th graded component L^0 is special because the adjoint action preserves each graded component L^i (indeed, this is related to the concept of an inner grading, which is a grading of the Lie algebra coming from the adjoint action of some element of the Lie algebra).

Definition 4.4.8. A dgLa is **ad^0-nilpotent** if for all i, the image $\mathrm{ad}(L^0) \to \mathrm{End}(L^i)$ is contained in a nilpotent associative Lie subalgebra.

A nilpotent dgLa is ad^0-nilpotent. But we don't need the entirety of our dgLa to be nilpotent, only really L^0. **If a dgLa L is ad^0-nilpotent then L^0 is nilpotent, which is enough to ensure that $\exp(L^0)$ is well-defined.** However, the converses of these statements are false.

The reason why we want ad^0-nilpotent dgLas is that we want the adjoint action to be a group action on each graded component L^i, which means we need $\mathrm{ad}(L^0) \subset \mathrm{End}(L^i)$ to actually land in the subset $\mathrm{Aut}(L^i)$ of invertible elements; this is guaranteed by Proposition 4.4.5.

Definition 4.4.9. Let L be an ad^0-nilpotent dgLa. We define the group homomorphisms $\mathrm{Ad}^i : \exp(L^0) \to GL(L^i)$ for all $i \in \mathbb{Z}$ by

$$\mathrm{Ad}^i(e^X) = \sum_{k \geq 0} \frac{(\mathrm{ad}\, X)^k}{k!}.$$

Concretely, we have the action of $\exp(L^0)$ on L^i by

$$e^X \cdot y = 1 + [X, Y] + \frac{1}{2!}[X, [X, Y]] + \frac{1}{3!}[X, [X, [X, Y]]] + \cdots .$$

This is precisely the reason why the group structure $\exp(L^0)$ is convenient for us: we have $\mathrm{Ad}^i(e^X) \circ \mathrm{Ad}^i(e^Y) = \mathrm{Ad}^i(e^{X*Y})$, **hence defining a group action $\exp(L^0) \curvearrowright L^i$.**

The most important definition in this subsection is:

Definition 4.4.10. Let L be a dgLa with differential δ. The **Maurer-Cartan equation** of L is

$$\delta(x) + \frac{1}{2}[x, x] = 0$$

for $x \in L^1$. Solution $x \in L^1$ are called **Maurer-Cartan elements of** L.

Remark 4.4.11. Note that $[x, x] \neq 0$! Recall that the bracket is **graded** so in fact for $x \in L^1$, then $[x, x] = (-1)^{1\cdot 1+1}[x, x] = [x, x]$, hence this can be nonzero.

Definition 4.4.12. Given a dgLa (L, δ) we define a new dgLa $(\mathrm{aff}(L), \partial)$ by $\mathrm{aff}(L) = L \oplus k\{d\}$ where d is a formal symbol of degree 1 mimicking δ in L. The bracket is given by $[x + vd, y + wd]_{\mathrm{aff}(L)} = [x, y]_L + v\delta(y) + (-1)^{|x|}w\delta(x)$, and $\partial(x + vd) = \delta(x)$. The map aff is functorial. As a result, we have an **affine embedding** $\varphi : L^1 \to \mathrm{aff}(L)^1$, $\varphi(x) = x + d$. Then

$$x \text{ is a Maurer-Cartan element of } L \iff [\varphi(x), \varphi(x)]_{\mathrm{aff}(L)} = 0.$$

If furthermore L is ad^0-nilpotent, then Ad^1 induces an action called the **gauge action** of $\exp(L^0)$ on the set of Maurer-Cartan elements by the affine embedding, given by $e^x \cdot y = \varphi^{-1}(\mathrm{Ad}^1(e^x)(\varphi(y)))$.

Whew, that was a heavy subsection. Let's see why we need all this machinery and setup for our original goal: to study deformation theory using dgLas.

4.5 Deformation functors and dgLas

The main idea is that **deformations (of say, a scheme) are in bijection with Maurer-Cartan elements modulo Gauge action**. Let's see it played out concretely in the following example. Let me remark that **this example is extremely useful and illuminating**, so if you've been skimming, you should at least carefully go through this example!

Example 4.5.1. Let V be a finite cochain complex of k-vector spaces:

$$0 \to V^1 \xrightarrow{\partial^2} V^2 \xrightarrow{\partial^3} \cdots \xrightarrow{\partial^n} V^n \to 0.$$

For $A \in \mathsf{Art}_k$, then a **deformation** of V over A is just a cochain complex

$$0 \to V^1 \otimes_k A \xrightarrow{\gamma^2} V^2 \xrightarrow{\gamma^3} \cdots \xrightarrow{\gamma^n} V^n \otimes_k A \to 0$$

such that if we quotient by $\mathfrak{m}_A$, we recover the original complex V via the canonical isomorphism $V^i \otimes_k k \cong V^i$.

Aside: a priori, a deformation of V over A is just any cochain complex of A-modules such that $\otimes_A A/\mathfrak{m}_A$ returns the cochain complex V of k-modules. But the fact that we literally need to recover each V^i, not just a quasi-isomorphic complex, implies that we need a deformation of V^i over A, and the only deformation is $V^i \otimes_k A$. This can be understood via the fact that deformations of smooth affine schemes are trivial, and here V^i is an affine space over k. The only difference, then, is that we may choose differential map on $0 \to V^1 \otimes_k A \to V^2 \otimes_k A \to \cdots \to V^n \otimes_k A \to 0$ which is not simply $\partial \otimes_k \mathrm{id}_A$, yet they agree upon quotient by $\mathfrak{m}_A$.

An **isomorphism of deformations** is an isomorphism of cochain complexes such that when we quotient by $\mathfrak{m}_A$, the induced map $V \to V$ is the identity map. We wish to study the deformation functor

$$\mathrm{Def}_V : \quad A \mapsto \{\text{deformations of } V \text{ over } A\}/\{\text{isomorphism}\}.$$

More specifically, we wish to study this functor by constructing a dgLa.

Define the cochain complex $H(V)$ by $H(V)^k = \mathrm{Hom}_k(V, V[k])$ (note that this is NOT the same as $\mathrm{Hom}^\bullet(V,V)$). It turns out there's a canonical dgLa structure on $H(V)$, given by $[f,g] = fg - (-1)^{(\deg f)(\deg g)}gf$ and differential $d(f) = [\partial, f] = \partial f - (-1)^{\deg f} f\partial$. The claim is that the dgLa $H(V)$ should somehow give us the deformation functor Def_V.

First, the definition of a deformation $(V \otimes_k A, \delta)$ is that when we take it modulo $\mathfrak{m}_A$, we recover (V, ∂). This means that $\delta - \partial \in \mathrm{Hom}^1(V,V) \otimes \mathfrak{m}_A$. Letting $\xi := \delta - \partial$, then the condition that $\delta^2 = 0$ translates to the condition $d(\xi) + \frac{1}{2}[\xi, \xi] = 0$: this is precisely the condition that ξ is a Maurer-Cartan element of $H(V) \otimes_k \mathfrak{m}_A$! Thus

$$\{\text{deformations of } V \text{ over } A\} \longleftrightarrow \{\text{Maurer-Cartan elements of } H(V) \otimes_k \mathfrak{m}_A\}.$$

It only remains to figure out what "isomorphic deformation" means in terms of the Maurer-Cartan elements. Suppose we have an isomorphism of deformations $\psi : (V \otimes_k A, \delta_1) \to (V \otimes_k A, \delta_2)$. A more difficult observation is that $\psi = \exp(a)$ for some $a \in H^0(V) \otimes_k \mathfrak{m}_A$. Upon closer reflection (which is to say, just believe me), we see that this is precisely the condition that $\xi_1 = \delta_1 - \partial$ and $\xi_2 = \delta_2 - \partial$ are related by a gauge action. Therefore Def_V is isomorphic to the group of Maurer-Cartan elements of $H(V) \otimes_k \mathfrak{m}_A$ modulo gauge action.

With this illuminating example at hand, we set about generalizing this to arbitrary dgLas. The intuition is that **deformations (not up to isomorphism) are parametrized by Maurer-Cartan elements lying in $L^1 \otimes \mathfrak{m}_A$**, and **isomorphic deformations are precisely those Maurer-Cartan elements linked by gauge group action.**

Definition 4.5.2. Fix L a dgLa over k. The **Maurer-Cartan functor** $MC_L : \mathsf{Art}_k \to \mathsf{Set}$ is defined by

$$MC_L : \quad A \mapsto \{x \in L^1 \otimes \mathfrak{m}_A : \quad \delta(x) + \frac{1}{2}[x,x] = 0\}.$$

The **gauge group functor** $G_L : \mathsf{Art}_k \to \mathsf{Grp}$ is defined by

$$G_L : \quad A \mapsto \exp(L^0 \otimes \mathfrak{m}_A).$$

Finally, define the **deformation functor assigned to** L $\mathrm{Def}_L : \mathsf{Art}_k \to \mathsf{Set}$ by

$$\mathrm{Def}_L(A) = MC_L(A)/G_L(A).$$

Remark 4.5.3. Recall that $G_L(A)$ acts on $MC_L(A)$ by the gauge action.

Proposition 4.5.4. *Fix L a dgLa. The gauge group functor G_L is a smooth homogeneous deformation functor. The Maurer-Cartan functor MC_L is a homogeneous deformation functor. If L is abelian, then MC_L is smooth.*

Finally, Def_L *is a deformation functor, but not necessarily homogeneous.*

Theorem 4.5.5.

- *If $H^2(L) = 0$, then MC_L and* Def_L *are smooth deformation functors.*
- *$H^2(L)$ is a complete obstruction space for* Def_L*. (See [ABCW15] for the precise construction.)*

There are more results identify cohomology groups as tangent spaces for these functors, as well as relative results induced by maps $f : L \to N$ of dgLas. For example,

Theorem 4.5.6. *If $H^0(f)$ is surjective, $H^1(f)$ is bijective, and $H^2(f)$ is injective, then the induced map* $\mathrm{Def}_L \to \mathrm{Def}_N$ *is an isomorphism.*

In particular, ***quasi-isomorphic dgLas give rise to isomorphic deformation functors****.*

Remark 4.5.7. The condition of being quasi-isomorphic is rather strong. There's a much weaker condition called L_∞-equivalence which leads to the same conclusion.

Remark 4.5.8. Let me clarify that H^2 is indeed a complete obstruction space, but **it is not necessarily universal** (see Definition 4.2.6), and therefore even when $f : L \to N$ satisfies the hypotheses of Theorem 4.5.6, it may be that $H^2(N) \subsetneq H^2(L)$ are **both** complete obstruction spaces, since they might both not be universal.

Now, unfortunately this is not the end of the story; for example, not every deformation functor comes from a dgLa. So the slogan that **dgLas control deformation theory of schemes** (in characteristic 0) is not entirely true. The point, however, is that in most "reasonable" cases, it's still true, and therefore **we can study deformation theory of a scheme by carefully choosing a dgLa whose deformation functor gives us the deformations of our scheme**. In practice, this could be rather difficult (and may not even exist), but to convince you that it's not total nonsense, we'll at least give the following examples.

Example 4.5.9. Let X be a smooth variety. Choose $\mathcal{U}$ an affine open cover of X, and let $\mathcal{T}_X$ be the tangent sheaf on X. Then the deformation theory of X is controlled by the dgLa $\check{C}(\mathcal{U}, \mathcal{T}_X)$, the Čech complex of $\mathcal{T}_X$ over $\mathcal{U}$, with differential the Čech differential and bracket the bracket of vector fields. (Note that we can choose the affine open cover, as any choice gives us a quasi-isomorphic complex, hence dgLa as well.)

Put more precisely, $\mathrm{Def}_X \cong \mathrm{Def}_{\check{C}(\mathcal{U}, \mathcal{T}_X)}$.

Compare this using Theorem 4.5.6 to the classical result: $H^0(X, \mathcal{T}_X)$ computes automorphisms of X, $H^1(X, \mathcal{T}_X)$ controls first-order deformations of X, and $H^2(X, \mathcal{T}_X)$ contains the obstructions.

Example 4.5.10. Let $\mathcal{E}$ be a vector bundle on a smooth (complex) variety X. Then deformations of $\mathcal{E}$ are controlled by the dgLa $\check{C}(\mathcal{U}, \mathcal{E}nd(\mathcal{E}))$, which is the Čech complex of the sheaf $\mathcal{E}nd(\mathcal{E})$ over some affine open cover $\mathcal{U}$ (since any choice of cover gives us quasi-isomorphic dgLas), with bracket given by commutator of composition of endomorphisms, and differential given by Čech differential.

Put more precisely, $\mathrm{Def}_X \cong \mathrm{Def}_{\check{C}(\mathcal{U}, \mathcal{E}nd(\mathcal{E}))}$.

Compare this using Theorem 4.5.6 to the classical result: $H^0(X, \mathcal{E}nd(\mathcal{E}))$ computes automorphisms of $\mathcal{E}$, $H^1(X, \mathcal{E}nd(\mathcal{E}))$ controls first-order deformations of $\mathcal{E}$, and $H^2(X, \mathcal{E}nd(\mathcal{E}))$ contains the obstructions.

Example 4.5.11. Let $Z \subset X$ be an embedded subvariety. This example is a bit more complicated since the sheaf being used is the normal bundle $\mathcal{N}_Z$, which is a quotient of $\mathcal{T}_X$. The classical case is that first-order deformations are controlled by $H^0(Z, \mathcal{N}_Z)$, while obstructions are given by $H^1(Z, \mathcal{N}_Z)$ (there are no automorphisms since we want embedded deformations). It turns out there is still a dgLa controlling the deformation theory here, albeit more complicated: it is the homotopy fiber of the inclusion $\check{C}(\mathcal{U}, \mathcal{T}_Z \to \mathcal{T}_X)$, with bracket the homotopy bracket and differential the Čech differential.

4.6 Derived deformation theory

There are a few problems we still want to patch up. First, it's not clear whether every deformation functor is Def_L for some dgLa. Secondly, $H^2(L)$ is not necessarily a unviersal obstruction theory for Def_L. (Sometimes there are no obstructions at all, yet $H^2(L) \neq 0$.)

The answer is that we need to go to the nicer world of derived deformations. Then everything works as we hope.

First, we need to transfer everything over to dg-land.

Definition 4.6.1. Denote by dgArt_k the category of local Artinian dg-algebras over k, i.e. dg-algebras A over k such that $\dim A < \infty$ and $\mathfrak{m}_A$ is nilpotent.

Let $\pi : A \twoheadrightarrow B$ be a surjection in dgArt_k with kernel I. Then $0 \to I \to A \to B \to 0$ is **semismall** if $I \cdot \mathfrak{m}_A = 0$ and **acyclic** if I is an acyclic complex.

Definition 4.6.2. A functor $F : \mathsf{dgArt}_k \to \mathsf{Set}$ is a **deformation functor** if

- $F(k)$ is a one-element set,
- for every semismall $A \twoheadrightarrow B$ and every $C \to B$ in dgArt_k, the natural map $F(A \times_B C) \to F(A) \times_{F(B)} F(C)$ is surjective,
- for every $A < B \in \mathsf{dgArt}_k$, the natural map $F(A \times_k B) \to F(A) \times F(B)$ is an isomorphism,
- if $A \twoheadrightarrow B$ is acyclic and semismall then $F(A) \xrightarrow{\sim} F(B)$ is an isomorphism.

We can then define the Maurer-Cartan functor, gauge group functor, and deformation functor Def_L of a dgLa L in exactly the same way as before. These are indeed deformation functors in the above dg sense.

Definition 4.6.3. Let $k[n]$ denote the local Artinian dga which is a single copy of k concentrated in degree $-n$. For a deformation functor F define the graded vector space $T^\bullet F$ where $T^i F := F(k[i-1])$. We say $T^1 F$ is the **tangent space** and $T^2 F$ is the **obstruction space** to F.

Definition 4.6.4. Let Def_k^M be the category of derived deformation functors.

The category Def_k^M admits a model structure where the weak equivalences are the natural transformations $F \to G$ inducing isomorphisms $T^i F \to T^i G$ for all $i \in \mathbb{Z}$.

Let dgLie_k be the category of dgLas over k.

dgLie_k admits the projective model structure where the weak equivalences are quasi-isomorphisms and fibrations are levelwise surjections.

Theorem 4.6.5. *The functor* Def_L *is the universal deformation functor under* MC_L*, i.e. for every deformation functor* F*, then every map* $MC_L \to F$ *factors through* Def_L*.*

Proposition 4.6.6. *If* L *is a dgLa then* $T^i\mathrm{Def}_L = H^i(L)$*.*

Theorem 4.6.7. *For every dgLa* L*, then* Def_L *is a derived deformation functor. Conversely, every derived deformation functor* F *comes from a dgLa* L*.*

More precisely, $\mathrm{Def} : \mathsf{dgLie}_k \to \mathsf{Def}_k^M$ *induces an equivalence of their homotopy categories.*

Whew!

Unfortunately this is still not the end of the story (although it is the end of *our* story). The deformation functors are not left exact, so they don't sheafify and there's no global version. The solution, proposed by Hinich, is to stackify the deformation functors. Then Pridham-Lurie show that $\mathrm{Def} : \mathsf{dgLie}_k \to \mathsf{Def}_k^H$ is an equivalence of ∞-categories, and therefore an equivalence of homotopy categories as well.

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