Springer fibers

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1 Introduction

1.1 Overview

These notes are for my pre-talk for the Harvard-MIT Algebraic Geometry seminar on April 2, 2024, expanded to include more interesting details and examples for completeness. The goal of these notes is to introduce the reader to the objects known as Springer fibers, and discuss some of their representation-theoretic properties. Springer fibers are extremely interesting objects because they encode the representation theory of the Weyl group *geometrically*. Weyl groups themselves are very interesting due to their fundamental importance in Lie theory, which is rooted in geometry. Therefore, one might naively hope that Weyl groups might stand out from typical finite groups by having their representations appear geometrically as well. This turns out to be true - it's known as the Springer correspondence - and is the springboard to geometric representation theory.

1.2 More details

These notes are *very* heavily based on [CG97, §3]; I strongly recommend you to read this chapter for more details. It's an absolutely wonderful book and paints a beautiful picture of Springer fibers and their place in representation theory. Another nice article which gives an overview of Springer fibers (and more) is [Yun16], and I referenced this often for concrete examples. In these notes, I don't aim to give complete details, nor full proofs; instead, I'll try to emphasize the most important ideas. It goes without saying that none of this is original.

1.3 Useful background

The main background the reader should have is being comfortable with the basic notion of Lie algebras, especially structure theory of semisimple Lie algebras, Cartan subalgebras, and Borel subalgebras. In a pinch, however, understanding just GL_n and SL_n should be sufficient. Basic algebraic geometry is useful to understand the maps we describe: often I'll write them set-theoretically for ease of understanding, but they are always maps of algebraic varieties. Basic algebraic topology is also useful, mainly to understand the basic uses of Borel-Moore homology and singular cohomology.

1.4 Setup and notation

We'll mainly work over the field $\mathbb{C}$. Let $\mathfrak{g}$ be a simple Lie algebra, therefore classified by some Dynkin diagram. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. There are many such choices of a Lie group, but every such group is a quotient of unique connected, simply-connected Lie group by a finite subgroup of the center. In our case the action of G on $\mathfrak{g}$ is always the *adjoint* action; since the action always factors through the center, every choice of a Lie group G gives us the same adjoint action on $\mathfrak{g}$.

At times, it's useful to fix a Cartan $\mathfrak{g} \subset \mathfrak{g}$, corresponding to a maximal torus $\mathrm{T} \subset \mathrm{G}$, and a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$, corresponding to a Borel subgroup $\mathrm{B} \subset \mathrm{G}$.

2 Preliminaries

Here is a quick summary of important preliminaries that we will use. For a more thorough treatment, see [CG97], especially the end of §2 and the beginning of §3. We adopt the setup and notation from §1.4.

2.1 Semisimple and regular elements

Recall that $\mathfrak{g}$ is a semisimple Lie algebra. Let $Z_{\mathfrak{g}}(x)$ denote the centralizer in $\mathfrak{g}$ of x. It's known that for any $x \in \mathfrak{g}$, there's the lower bound $\dim Z_{\mathfrak{g}}(x) \geq \operatorname{rk} \mathfrak{g} = \dim \mathfrak{h}$.

Definition 2.1 (regular): An element $x \in \mathfrak{g}$ is **regular** of $\dim Z_{\mathfrak{g}}(x) = \operatorname{rk} \mathfrak{g}$.

Remark 2.2: There's two ways to "centralize" $x \in \mathfrak{g}$ (or more generally, any set): there's the stabilizer subgroup in G acting by adjoint (conjugation), denoted by G_x, and there's the Lie subalgebra $Z_{\mathfrak{g}}(x)$ of elements whose adjoint action kills x. It's not hard to see that $Z_{\mathfrak{g}}(x)$ is the Lie algebra of G_x.

Definition 2.3 (semisimple): An element $x \in \mathfrak{g}$ is **semisimple** if $\operatorname{ad} x$ is diagonalizable.

We can think of semisimple as belonging to a Cartan subalgebra (this is literally true). Elements which are *both* semisimple and regular form a G-stable open dense subset of $\mathfrak{g}$; their complement is the closed subvariety cut out by a single polynomial.

2.2 Abstract Cartan and abstract Weyl group

For a choice of maximal torus $\mathsf{T} \subset \mathsf{G}$, we can realize the Weyl group explicitly as $N_{\mathsf{G}}(\mathsf{T})/\mathsf{T}$. However, this requires us to choose a maximal torus. These are useful in practice, but we need an abstract way to realize the Weyl group so that we are not forced to make a choice at every step and keep track of all of these choices.

Recall that $\mathfrak{g}$ is constructed abstractly from a (semisimple) root system $\mathcal{R}$ in a complex vector space $\mathfrak{H}$. This means that $\mathcal{R}$ is a finite subset of $\mathfrak{H}^*$, the dual space of $\mathfrak{H}$, and for each root $\alpha \in \mathcal{R}$ we are given an element $\alpha^\vee \in \mathfrak{H}$, called the coroots. The data then needs to satisfy some conditions. Now for $\alpha \in \mathcal{R}$, we have an associated reflection $s_\alpha : \mathfrak{H} \to \mathfrak{H}$ given by $s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \cdot \alpha$.

Definition 2.4 (abstract Weyl group): The **abstract Weyl group** $\mathbb{W}$ is the (finite) group generated by the reflections s_α.

Fix once and for all a base of $\mathcal{R}$, i.e., a set of simple roots $S \subset \mathcal{R}$. Then $\mathbb{W}$ is generated by $\{s_\alpha \mid s \in S\}$, the *simple reflections*. (Concretely, $\mathbb{W}$ is a Coxeter group, so each s_α has order two and the braiding relations are given by the angles between the simple roots.)

To relate $\mathbb{W}$ to each W_{T} is akin to "choosing an orientation" on G. If G were the sphere S^2, then I think of choosing T as choosing some great circle as the equator, and choosing B is choosing which hemisphere is the northern one. In $\mathbb{W}$, the simple roots $S \subset \mathcal{R}$ already choose the "northern hemisphere" for us.

To construct this identification is slightly delicate, so I'll spell it out slowly (with emphasis on especially important words).

Lemma 2.5: Any two quotients $\mathfrak{b}/[\mathfrak{b},\mathfrak{b}]$, $\mathfrak{b}'/[\mathfrak{b}',\mathfrak{b}']$ are *canonically isomorphic*.

Proof. Any two Borel subalgebras are G-conjugate, so we always have an isomorphism $\mathfrak{b} \xrightarrow{\sim} \mathfrak{b}'$ given by $\mathfrak{b}' = g\mathfrak{b}g^{-1}$. But we have a whole coset $g\mathsf{B}$ to give us these maps. The key is that when we pass to the *quotient*, each element of the coset $g\mathsf{B}$ induces *exactly the same map* $\mathfrak{b}/[\mathfrak{b},\mathfrak{b}] \xrightarrow{\sim} \mathfrak{b}'/[\mathfrak{b}',\mathfrak{b}']$, thus making the isomorphism *canonical.* □

So now we **identify all quotients** $\mathfrak{b}/[\mathfrak{b},\mathfrak{b}]$ (running over all Borel subalgebras $\mathfrak{b}$) and call this vector space the "abstract Cartan" $\mathfrak{H}$. Note that this is *not* a subalgebra of $\mathfrak{g}$, which is why it is called "abstract"!

In order for this to make sense, we need to relate this "abstract Cartan" $\mathfrak{H}$ with some concrete Cartan $\mathfrak{h} \subset \mathfrak{g}$, so that we can turn this abstract Cartan into an abstract construction of $\mathfrak{g}$ through root data which also agrees with the many concrete constructions (through picking some concrete $\mathfrak{h}$). Currently, $\mathfrak{H}$ is merely *some* very special abstract vector space, and we need to specify abstract roots and choose a subset to be the simple abstract roots, then check its compatibility with all of the concrete constructions.

The idea is that every choice of $\mathfrak{h} \subset \mathfrak{g}$ gives us a root system, and every choice of $\mathfrak{b} \supset \mathfrak{h}$ gives us a specification of simple roots, thus giving us all of the data of $\mathfrak{H}, \mathcal{R}, S, \mathbb{W}$, and they all end up giving compatible identifications. (For this reason, we can view choosing $\mathfrak{h}$ as the "plane" on which we live on, for example the choice of equator in S^2, and choosing $\mathfrak{b}$ as the choice of northern hemisphere.)

First, any choice of Cartan $\mathfrak{h} \subset \mathfrak{g}$ immediately gives us the data of a root system: the vector space is $\mathfrak{h}^*$, and the roots $\mathcal{R}_{\mathfrak{g},\mathfrak{h}} \subset \mathfrak{h}^*$ are exactly the weights of the adjoint $\mathfrak{h}$-action on $\mathfrak{g}$. (The coroots can be defined using the Killing form.) This root system is not complete, however: it lacks a choice of simple roots. To get that, we need to choose a Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$; the positive roots are exactly the weights of the adjoint $\mathfrak{h}$-action on $\mathfrak{b}$. This gives us all of the data we need: we have the vector space $\mathfrak{h}^*$, the root system $\mathcal{R}_{\mathfrak{g},\mathfrak{h}}$, and a choice of simple roots $S_{\mathfrak{b}}$; indeed, this is the classical construction.

Remark 2.6: To reiterate the point: a choice of maximal torus $\mathsf{T} \subset \mathsf{G}$ is enough to define the Weyl group W_{T}; however, in order to have a specified set of simple reflections as the generators, we need to choose a Borel B. So if we don't care about the "orientation" of W_{T} and only care about it as a group, then we can just write W_{T} and ignore the choice of B.

The first step is to compare the Cartans. The chain of maps $\mathfrak{h} \hookrightarrow \mathfrak{b} \twoheadrightarrow \mathfrak{b}/[\mathfrak{b},\mathfrak{b}]$ gives us an isomorphism $\mathfrak{h} \xrightarrow{\sim} \mathfrak{H}$, which induces an isomorphism $\mathfrak{h}^* \xrightarrow{\sim} \mathfrak{H}^*$. Thus, we can define the roots $\mathcal{R}$ of $\mathfrak{H}$ to be the image of $\mathcal{R}_{\mathfrak{g},\mathfrak{h}}$ under this identification. Next, the choice of Borel $\mathfrak{b} \supset \mathfrak{h}$ gives us a choice of simple roots $S_{\mathfrak{b}} \subset \mathcal{R}_{\mathfrak{g},\mathfrak{h}}$, and therefore the isomorphism $\mathfrak{h} \xrightarrow{\sim} \mathfrak{H}$ gives us a choice of simple roots $S \subset \mathcal{R}$. Lemma 2.2 tells us that *any* choice of Borel subalgebra containing this fixed $\mathfrak{h}$ would induce the exact same isomorphisms $\mathfrak{h} \xrightarrow{\sim} \mathfrak{H}$ and roots $\mathcal{R}_{\mathfrak{g},\mathfrak{h}} \xrightarrow{\sim} \mathcal{R}$, and furthermore *also specify the same exact simple roots in* $\mathfrak{H}^*$.

Finally, we need to say how the Weyl groups get related. A choice of $\mathfrak{h}$, equivalently T, gives us a root system, which is enough to define the Weyl group $W_{\mathsf{T}} := N_{\mathsf{G}}(\mathsf{T})/\mathsf{T}$. The choice of $\mathfrak{h}$ also gives us an isomorphism $\mathfrak{h} \xrightarrow{\sim} \mathfrak{H}$, which does not depend on any Borel subalgebra; this induces an isomorphism $W_{\mathsf{T}} \xrightarrow{\sim} \mathbb{W}$. However, $\mathbb{W}$ is equipped with more structure than simply an abstract group: it has a specified set of generators, the simple reflections. The choice of $\mathfrak{b} \supset \mathfrak{h}$ gives the simple generators of W_{T}, and since the isomorphism $\mathcal{R}_{\mathfrak{g},\mathfrak{h}} \xrightarrow{\sim} \mathcal{R}$ induced by $(\mathfrak{h},\mathfrak{b})$ takes the simple roots $S_{\mathfrak{b}}$ to the simple roots S, the isomorphism $W_{\mathsf{T}} \xrightarrow{\sim} \mathbb{W}$ must also take the simple reflections to the simple reflections.

So we've shown that the abstract Cartan $\mathfrak{H} = \mathfrak{b}/[\mathfrak{b},\mathfrak{b}]$, along with the abstract root system $\mathcal{R}$, choice of simple reflections S, and abstract Weyl group $\mathbb{W}$ can all be related to a concrete set W_{T} and choice of simple generators arising from the pair $(\mathfrak{h},\mathfrak{b})$. It only remains to understand how the isomorphisms $W_{\mathsf{T}} \xrightarrow{\sim} \mathbb{W}$ are compatible as $(\mathfrak{h},\mathfrak{b})$ vary. The answer is essentially that as pairs $(\mathfrak{h},\mathfrak{b})$ vary via G-conjugation, the W_{T} also changes by conjugation. Explicitly, for a pair $\mathfrak{h} \subset \mathfrak{b}$, put $W_{\mathsf{T},\mathsf{B}}$ for the concrete Weyl group $N_{\mathsf{G}}(\mathsf{T})/\mathsf{T}$ *with specified simple reflections*. We have the isomorphism $\phi_{\mathfrak{h},\mathfrak{b}} : W_{\mathsf{T},\mathsf{B}} \xrightarrow{\sim} \mathbb{W}$ constructed as above (here we view $\mathbb{W}$ as the abstract Weyl group equipped with the data of simple reflections as generators). Then for another pair $(g\mathfrak{h}g^{-1}, g\mathfrak{b}g^{-1})$, we have another concrete Weyl group $W_{g\mathsf{T}g^{-1},g\mathsf{B}g^{-1}} = N_{\mathsf{G}}(g\mathsf{T}g^{-1})/g\mathsf{T}g^{-1}$, *again with specified simple reflections*, along with an isomorphism $\phi_{g\mathfrak{h}g^{-1},g\mathfrak{b}g^{-1}} : W_{g\mathsf{T}g^{-1},g\mathsf{B}g^{-1}} \xrightarrow{\sim} \mathbb{W}$. Then we have a commutative diagram

$$\begin{tikzcd}
W_{\mathsf{T},\mathsf{B}} \arrow[dr, "\phi_{\mathfrak{h},\mathfrak{b}}", "\sim"'] \arrow[dd, "\mathrm{Ad}(g)"', "\sim"] & \\
& \mathbb{W} \\
W_{g\mathsf{T}g^{-1},g\mathsf{B}g^{-1}} \arrow[ur, "\sim", "\phi_{g\mathfrak{h}g^{-1},g\mathfrak{b}g^{-1}}"'] &
\end{tikzcd}$$

summarizing the compatibilities. In other words, **choosing T and B is choosing an orientation for which you want to view G through**, and the Weyl group moves accordingly and compatibly.

2.3 Flag variety

Definition 2.7 (flag variety): The **flag variety** of $\mathfrak{g}$ is the closed subvariety of the Grassmannian of $\dim \mathfrak{b}$-dimensional subspaces in $\mathfrak{g}$ formed by all solvable Lie subalgebras.

This is not a very clear definition, so let me give the much more common perspectives:

a) it's the moduli space (i.e., set) of all Borel subalgebras in $\mathfrak{g}$;

b) it's the moduli space (i.e., set) of all Borel subgroups in G;

c) it's isomorphic to G/B.

The first two are obviously equivalent; they're equivalent to the third by understanding that any Borel subalgebra is obtained from another by conjugation by G, and the stabilizer is precisely the corresponding Borel subgroup B. Explicitly, fixing a pair of corresponding Borels $\mathfrak{b} \leftrightarrow \mathsf{B}$, the isomorphism $\mathsf{G}/\mathsf{B} \xrightarrow{\sim} \mathcal{B}$ is realized by $g \mapsto g\mathfrak{b}g^{-1}$.

The flag variety is a G-homogeneous projective variety, with G acting on Borels by conjugation, or equivalently, on G/B by left multiplication.

When $\mathfrak{g} = \mathfrak{sl}_n$, the flag variety has another interpretation. Each Borel subalgebra is uniquely associated to a **full flag** of $\mathbb{C}^n$, which is a sequence of subspaces $0 = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = \mathbb{C}^n$, where $\dim V_i = i$. For example, the Borel subalgebra corresponding to upper triangular matrices has the natural full flag given by the addition of the coordinate vectors, one at a time: $0 \subsetneq \langle e_1 \rangle \subsetneq \langle e_1, e_2 \rangle \subsetneq \cdots \subsetneq e_1, e_2, \ldots, e_n \rangle$. Therefore we can view the flag variety of $\mathfrak{sl}_n$ as the moduli space of full flags of $\mathbb{C}^n$.

2.4 Bruhat decomposition

Fix some Borel $\mathrm{B} \subset \mathrm{G}$ corresponding to $\mathfrak{b} \subset \mathfrak{g}$, along with a maximal torus $\mathrm{T} \subset \mathrm{B}$. We can concretely realize the Weyl group with the alignment (T, B) by $W_\mathrm{T} := N_\mathrm{G}(\mathrm{T})/\mathrm{T}$. The punchline is that

Theorem 2.8 (Bruhat decomposition): The double cosets (given by left and right multiplication) $\mathrm{B}\backslash\mathrm{G}/\mathrm{B}$ are in bijection with W_T.

As an immediate corollary, W_T is in bijection with B-orbits on the flag variety $\mathcal{B}$, identifying $\mathcal{B} \simeq \mathrm{G}/\mathrm{B}$. Finally, we have a bijection

$$\{\text{B-orbits on } \mathcal{B}\} \leftrightarrow \{\text{G-diagonal orbits on } \mathcal{B} \times \mathcal{B}\}, \quad \mathrm{B} \cdot \mathfrak{b}' \mapsto \mathrm{G} \cdot (\mathfrak{b}, \mathfrak{b}').$$

Note that this last one *doesn't depend on* T, B. Therefore, we have a parametrization of diagonal G-orbits in $\mathcal{B} \times \mathcal{B}$ by $\mathbb{W}$. Denote the orbits by Y_w for $w \in \mathbb{W}$. These will appear in §4.6.

2.5 The nilpotent cone

Definition 2.9 (nilpotent element): A **nilpotent element** is an element of $\mathfrak{g}$ which acts nilpotently on every finite-dimensional $\mathfrak{g}$-module.

There are several other equivalent characterizations of it (for example, see [CG97, §3]), but one simple way to think about it is to embed $\mathfrak{g} \hookrightarrow \mathfrak{gl}_n$, and then the nilpotent elements are literally the elements which matrices (in $\mathfrak{gl}_n$) are nilpotent.

Definition 2.10 (nilpotent cone): The **nilpotent cone** $\mathcal{N}$ is the cone subvariety of all nilpotent elements in $\mathfrak{g}$.

The nilpotent cone is singular at exactly one point: the origin $0 \in \mathfrak{g}$.

Example 2.11 ($\mathcal{N}(\mathfrak{sl}_2)$): Let $\mathfrak{g} = \mathfrak{sl}_2 = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a, b, c \in \mathbb{C} \right\}$. An element is nilpotent iff it has determinant 0. Therefore we have the explicit description

$$\mathcal{N} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid -a^2 - bc = 0 \right\},$$

so $\mathcal{N}$ is a quadratic cone in $\mathbb{C}^3$.

2.6 Borel-Moore homology

Borel-Moore homology is another type of homology theory for "reasonable" spaces. For example, every algebraic variety over $\mathbb{C}$ or $\mathbb{R}$ satisfies this. The Borel-Moore homology of X is denoted by $H_\bullet^{BM}(X)$. For a much more thorough summary, see [CG97, §2.6]; here, I'll *very* briefly summarize the important facts.

Borel-Moore homology still has many of the important properties that we're familiar with from usual algebraic topology: proper pushforwards, long exact sequences, intersection pairings, Poincaré duality, Künneth formula, restriction with supports, smooth pullback, projection formula, even specialization.

One of the key selling points of Borel-Moore homology is the existence of a fundamental class. In singular homology, a smooth oriented manifold X has a fundamental class iff X is compact. But in Borel-Moore homology, there is always a fundamental class in the top Borel-Moore homology group. In fact, every irreducible component of X has a fundamental class, and the top Borel-Moore homology group has a basis given by the fundamental classes of the top-dimensional irreducible components of X.

2.7 Convolution

The most important part of Borel-Moore homology (for our purposes) is **convolution**. (See [CG97, §2.7] for more details.) Let M_1, M_2, M_3 be connected oriented smooth manifolds, and take $Z_{1,2} \subset M_1 \times M_2$ and $Z_{2,3} \subset M_2 \times M_3$ to be closed subsets. Define the "composition"

$$Z_{1,2} \circ Z_{2,3} := \{(m_1, m_3) \in M_1 \times M_3 \mid \exists\, m_2 \in M_2 \text{ such that } (m_1, m_2) \in Z_{1,2},\ (m_2, m_3) \in Z_{2,3}.\}$$

The way to think about this is to think of $Z_{i,j}$ as "multi-valued maps from M_i to M_j," and think of composition as composition of these "multi-valued maps." As an example, if $f : M_1 \to M_2$ and $g : M_2 \to M_3$ were smooth maps, then $\mathrm{Graph}(f) \circ \mathrm{Graph}(g) = \mathrm{Graph}(g \circ f)$.

Let $p_{i,j} : M_1 \times M_2 \times M_3 \to M_i \times M_j$ be the projection map. We **assume** that

$$p_{1,3} : \quad p_{1,2}^{-1}(Z_{1,2}) \cap p_{2,3}^{-1}(Z_{2,3}) \to M_1 \times M_3 \quad \text{is proper.}$$

Definition 2.12 (convolution): Let $d := \dim_{\mathbb{R}} M_2$. The **convolution** map

$$H_i^{BM}(Z_{1,2}) \times H_j^{BM}(Z_{2,3}) \to H_{i+j-d}^{BM}(Z_{1,2} \circ Z_{2,3}), \quad (c_{1,2}, c_{2,3}) \mapsto c_{1,2} \star c_{2,3},$$

is defined by

$$c_{1,2} \star c_{2,3} = (p_{1,3})_* \left((c_{1,2} \boxtimes [M_3]) \cap [M_1] \boxtimes c_{2,3})\right).$$

To summarize, **whenever we have varieties X, Y, Z satisfying $X \circ Y = Z$, then we get an induced map in Borel-Moore homology $H_\bullet^{BM}(X) \otimes H_\bullet^{BM}(Y) \to H_\bullet^{BM}(Z)$.**

An easy, but important, consequence is that when $X \circ X = X$, then the convolution map turns $H_\bullet^{BM}(X)$ into an associative algebra. For example, this happens when we have a proper map $\pi : M \to N$ for M a smooth complex manifold, and $X = M \times_N M$. The unit in $H_\bullet^{BM}(X)$ is given by the fundamental class of the diagonal $M_\Delta \subset X$.

3 Extremely succinct summary

In case you don't want to read the rest of the note, I'll summarize the notes here, *extremely* succinctly.

3.1 The basic objects

Definition 3.1 (Springer resolution): Let $\widetilde{\mathfrak{g}} := \{(x, \mathfrak{b}) \in \mathfrak{g} \times \mathcal{B} \mid x \in \mathfrak{b}\}$. Define the map $\mu : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ by $(x, \mathfrak{b}) \mapsto x$.
Define $\widetilde{\mathcal{N}} := \mu^{-1}(\mathcal{N}) = \{(n, \mathfrak{b}) \in \mathcal{N} \times \mathcal{B} \mid n \in \mathfrak{b}\}$ to be the part lying over the nilpotent cone. The **Springer resolution** is the restriction of the map μ to $\widetilde{\mathcal{N}}$:

$$\mu : \widetilde{\mathcal{N}} \to \mathcal{N}, \quad (n, \mathfrak{b}) \mapsto n,$$

and as the name suggests, it is a resolution of singularities.

Definition 3.2: To $x \in \mathfrak{g}$, define $\mathcal{B}_x := \mu^{-1}(x)$ to be the **Springer fiber** at $x \in \mathfrak{g}$. We regard it as a subset of $\mathcal{B}$.

Springer fibers are our main object of interest in these notes. We also need to know:

Definition 3.3 (Steinberg variety): The **Steinberg variety** is defined to be

$$Z := \widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}} = \{(x, \mathfrak{b}, \mathfrak{b}') \in \mathcal{N} \times \mathcal{B} \times \mathcal{B} \mid x \in \mathfrak{b} \cap \mathfrak{b}' \cap \mathcal{N}\}.$$

3.2 Geometric properties

First, an observation: as varieties up to isomorphism, Springer fibers depend only on the G-orbit that they live in.

It turns out that Springer fibers are connected and pure-dimensional.

The Steinberg variety is also connected and pure-dimensional, of dimension $2\dim\mathcal{B}$. Furthermore, its irreducible components are indexed by $w \in \mathbb{W}$, so its top Borel-Moore homology is a $|\mathbb{W}|$-dimensional vector space.

3.3 Representation theory

The main theorem converting algebra into geometry is that the top Borel-Moore homology group of Z, denoted by $H(Z)$, is an associative algebra under convolution, isomorphic to the group algebra of $\mathbb{W}$. Furthermore, the top Borel-Moore homology groups of the Springer fibers, denoted by $H(\mathcal{B}_x)$, naturally carry an action of a left (and right, but we won't need it here) $H(Z)$-module, hence is a $\mathbb{W}$-representation. Noting that the stabilizer group G_x acts on $\mathcal{B}_x$, we also have an action by G_x on $H(\mathcal{B}_x)$ factoring through the component group $\pi_0(\mathrm{G}_x)$, and this action commutes with the $H(Z)$-action. Since $\pi_0(\mathrm{G}_x)$ is a finite group, we can decompose $H(\mathcal{B}_x)$ with respect to its $\pi_0(\mathrm{G}_x)$-module structure (as direct sum of irreducible representations tensored with multiplicity spaces), and the multiplicity spaces will end up being the irreducible $\mathbb{W}$-representations. This turns out to completely enumerate all of the irreducible $\mathbb{W}$-representations, and so we have a very explicit geometric construction of irreducible $\mathbb{W}$-representations: this is called the Springer correspondence (5.2).

4 The geometry of Springer fibers

4.1 Grothendieck simultaneous resolution

Definition 4.1: Define the incidence variety $\widetilde{\mathfrak{g}} := \{(x, \mathfrak{b}) \in \mathfrak{g} \times \mathcal{B} \mid x \in \mathfrak{b}\}$.

We can use standard incidence variety procedures to understand this variety. First we have the canonical projections μ and π to $\mathfrak{g}$ and $\mathcal{B}$, respectively:

$$\begin{tikzcd} & \widetilde{\mathfrak{g}} \arrow[dl, "\mu"'] \arrow[dr, "\pi"] & \\ \mathfrak{g} & & \mathcal{B}. \end{tikzcd}$$

We are actually more interested in the (fibers of the) map μ, but the map π is easier to understand: for every $\mathfrak{b} \in \mathcal{B}$, it's clear that $\pi^{-1}(\mathfrak{b})$ is just the elements of $\mathfrak{b}$. It follows that π makes $\widetilde{\mathfrak{g}}$ a vector bundle over $\mathcal{B}$ whose fibers are Borel subalgebras. But there is more going on than that: both $\mathfrak{g}$ and $\mathcal{B}$ have an action of G given by conjugation (i.e., adjoint on $\mathfrak{g}$ and conjugation on $\mathcal{B}$), and this turns $\widetilde{\mathfrak{g}}$ into a G-equivariant vector bundle. More precisely,

Proposition 4.2: Fix a corresponding pair of Borel subalgebra and subgroup $\mathfrak{b}, \mathrm{B}$, and identify $\mathrm{G}/\mathrm{B} \xrightarrow{\sim} \mathcal{B}$. There's a G-equivariant isomorphism $\mathrm{G} \times^{\mathrm{B}} \mathfrak{b} \xrightarrow{\sim} \widetilde{\mathfrak{g}}$, which identifies the projection map to $\mathrm{G}/\mathrm{B} \xrightarrow{\sim} \mathcal{B}$ with the projection map π.

The exact isomorphism is given by $(g, x) \mapsto (gxg^{-1}, g\mathfrak{b}g^{-1})$.

Remark 4.3: The G-equivariant bundle $\mathrm{G} \times^{\mathrm{B}} \mathfrak{b}$ is defined to be the orbit-space of the trivial bundle $\mathrm{G} \times \mathfrak{b}$ under the free B-action given by $b \cdot (g, x) = (gb^{-1}, bxb^{-1})$.

Now that we have an understanding of $\widetilde{\mathfrak{g}}$, we can turn to the map μ. First, because μ factors as $\widetilde{\mathfrak{g}} \hookrightarrow \mathfrak{g} \times \mathcal{B} \to \mathfrak{g}$, and $\mathcal{B}$ is a projective variety, μ is proper. The fibers of this map give us the most important object in these notes.

Definition 4.4: To $x \in \mathfrak{g}$, define $\mathcal{B}_x := \mu^{-1}(x)$ to be the **Springer fiber** at $x \in \mathfrak{g}$.

So $\mathcal{B}_x = \{\mathfrak{b} \in \mathcal{B} \mid \mathfrak{b} \ni x\} \subset \mathcal{B}$. We will usually consider Springer fibers as a subvariety of the flag variety $\mathcal{B}$, i.e., identified with its image under π.

Remark 4.5: The most interesting Springer fibers are those over the nilpotent cone, i.e., when x is nilpotent, but there's no harm in considering non-nilpotents as well.

Springer fibers change as the "type" of element we choose in $\mathfrak{g}$ changes. To be more precise,

Note 4.6: Springer fibers in the same G-orbits of $\mathfrak{g}$ are isomorphic (as varieties). This is because for x, y in the same G-orbit, then there is some $g \in \mathrm{G}$ with $y = gxg^{-1}$, and correspondingly $\mathcal{B}_y = g\mathcal{B}_x g^{-1}$.

Example 4.7: The extreme cases are the most special point $x = 0$, and the open dense locus $\mathfrak{g}^{sr}$ of semisimple regular elements.

a) When $x = 0$, then $\mathcal{B}_0 = \mathcal{B}$. This is because every Borel subalgebra contains 0.

b) When $x \in \mathfrak{g}^{sr}$, then there are exactly $|\mathbb{W}|$ many points in $\mathcal{B}_x$, and in fact there's a canonical free $\mathbb{W}$-action on $\mathcal{B}_x$ (recall that the Borel subalgebras containing a fixed semisimple regular element are indexed by $\mathbb{W}$, by their "relative position").
This action is compatible with the entirety of $\mathfrak{g}^{sr}$, so that the projection $\mu^{-1}(\mathfrak{g}^{sr}) \to \mathfrak{g}^{sr}$ is a principle $\mathbb{W}$-bundle.

Example 4.8: When $\mathrm{G} = \mathrm{SL}_n$, then the flag variety $\mathcal{B}$ is the moduli space of full flags $0 = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = \mathbb{C}^n$. Then the Springer fiber $\mathcal{B}_e$ of a nilpotent element $e \in \mathcal{N}$ is exactly those flags for which $e(V_i) \subset V_{i-1}$ for all i.

The main diagram here is the following:

Proposition 4.9 (Grothendieck simultaneous resolution): We have natural maps $\rho : \mathfrak{g} \to \mathfrak{g}/\mathrm{G} \cong \mathfrak{H}/\mathbb{W}$ and $\pi : \mathfrak{H} \to \mathfrak{H}/\mathbb{W}$, which fit into the following commutative diagram:

$$\begin{tikzcd}
& \widetilde{\mathfrak{g}} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\
\mathfrak{g} \arrow[dr, "\rho"'] & & \mathfrak{H} \arrow[dl, "\pi"] \\
& \mathfrak{H}/\mathbb{W}. &
\end{tikzcd}$$

The restriction of this diagram to the regular locus is Cartesian.
Furthermore, for every $h \in \mathfrak{H}$, the map $\mu : \nu^{-1}(h) \to \rho^{-1}(h)$ is a resolution of singularities.

4.2 Springer resolution

In Proposition 4.1, specializing to $h = 0$, we obtain the following:

Definition 4.10 (Springer resolution): Let $\widetilde{\mathcal{N}} \coloneqq \nu^{-1}(0) = \{(n, \mathfrak{b}) \in \mathcal{N} \times \mathcal{B} \mid n \in \mathfrak{b}\}$. It's clear that $\rho^{-1}(0) = \mathcal{N}$. The **Springer resolution** is the restriction of μ to $\widetilde{\mathcal{N}}$:

$$\mu : \widetilde{\mathcal{N}} \to \mathcal{N},$$

and as the name suggests, it is a resolution of singularities.

We can understand $\widetilde{\mathcal{N}}$ as the pullback of $\widetilde{\mathfrak{g}}$ along the inclusion $\mathcal{N} \hookrightarrow \mathfrak{g}$:

$$\begin{tikzcd} \widetilde{\mathcal{N}} \arrow[r, hook] \arrow[d, "\mu"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathfrak{g}} \arrow[d, "\mu"'] \\ \mathcal{N} \arrow[r, hook] & \mathfrak{g}. \end{tikzcd}$$

There are several more concrete ways to understand it:

Proposition 4.11: Fix a Borel subgroup $\mathrm{B} \subset \mathrm{G}$ and a corresponding Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$, and let $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ be the unipotent subalgebra inside. We have

$$\widetilde{\mathcal{N}} \cong \mathrm{G} \times^{\mathrm{B}} \mathfrak{n} = \mathrm{G} \times^{\mathrm{B}} \mathfrak{b}^{\perp} \cong T^*\mathcal{B}.$$

In fact, the map $\pi : \widetilde{\mathcal{N}} \to \mathcal{B}$ is just the projection map $T^*\mathcal{B} \to \mathcal{B}$.

Proof. The first isomorphism follows immediately from the fact that $\widetilde{\mathfrak{g}} \cong \mathrm{G} \times^{\mathrm{B}} \mathfrak{b}$, and the fact that $\mathfrak{b} \cap \mathcal{N} = \mathfrak{n}$. The second equality follows from using the Killing form to identify $\mathfrak{g}^* \simeq \mathfrak{g}$, for which $\mathfrak{b}^{\perp}$ gets identified with $\mathfrak{n}$. The last equality follows from understanding the cotangent space at $\mathfrak{b} \in \mathcal{B}$ as the annihilator of $\mathfrak{b} \subset \mathfrak{g}$, which is $\mathfrak{b}^{\perp}$. □

Corollary 4.12: $\widetilde{\mathcal{N}}$ is smooth.

We can also easily compute $\dim \mathcal{N}$:

Corollary 4.13: $\mathcal{N}$ is irreducible of dimension $2 \cdot \dim \mathfrak{n}$.

Proof. First, since $\widetilde{\mathcal{N}} \cong T^*\mathcal{B} \twoheadrightarrow \mathcal{N}$, we can upper bound the dimension by $\dim T^*\mathcal{B} = 2 \dim \mathfrak{n}$. Also, since $\mathcal{N}$ is the fiber $\rho^{-1}(0)$, we know that the relative dimension $\dim \mathcal{N} - \dim\{0\} \geq \dim \mathfrak{g} - \dim \mathfrak{H}/\mathbb{W} = 2 \dim \mathfrak{n}$, which gives us the upper bound. □

4.3 Nilpotent orbits and important Springer fibers

We already saw in Example 4.1 the specific cases of Springer fibers for $x = 0$ and x a semisimple regular element. What about other cases?

As noted in 4.1, Springer fibers depend only on the G-orbit of the element. Therefore, we need to know the G-orbits of $\mathcal{N}$.

Proposition 4.14: There are finitely many G-orbits in $\mathcal{N}$.

We'll defer the proof of this to later; for now, this immediately lets us describe an important G-orbit:

Corollary 4.15: The regular nilpotent elements $\mathcal{N}^{reg}$ form the unique dense open G-orbit in $\mathcal{N}$.

Proof. Since there are finitely many G-orbits, there is a unique dense open orbit. It suffices to check that for $x \in \mathcal{N}^{reg}$, that $G \cdot x$ has dimension equal to $\mathcal{N}$. But we know that the dimension of the stabilizer $\dim \mathrm{G}_x = \dim Z_{\mathfrak{g}}(x) = \dim \mathfrak{h}$, so $\dim \mathrm{G} \cdot x = \dim \mathrm{G} - \dim \mathrm{G}_x = 2 \cdot \dim \mathfrak{n} = \dim \mathcal{N}$. □

Some more things which we know about the structure of G-orbits in $\mathcal{N}$:

a) Every G-orbit in $\mathfrak{g}$, therefore also in $\mathcal{N}$, is even-dimensional. This is because to $x \in \mathfrak{g}$, we can modify the Killing form κ to obtain an antisymmetric bilinear form $\beta_x(y, z) := \kappa(x, [y, z])$ on $\mathfrak{g}$; by the identity $\kappa(x, [y, z]) = \kappa([x, y], z)$ and non-degeneracy of κ, it follows that $\ker \beta_x = Z_{\mathfrak{g}}(x)$, hence β_x induces a non-degenerate antisymmetric bilinear form on $\mathfrak{g}/Z_{\mathfrak{g}}(x) = \mathrm{Lie}\, \mathrm{G} \cdot x$.

b) In $\mathcal{N}$, there is a unique dense open orbit and a unique closed orbit (consisting of the point 0). There are no orbits of dimension 1 or codimension 1, as above; however, there is a unique orbit of codimension 2 called the subregular orbit, as well as a unique smallest orbit larger than the zero orbit (but the dimension is not necessarily 2), called the minimal orbit.

Remark 4.16: There is quite a lot known about the combinatorics of nilpotent orbits; for example, see [CM93, Gu17]. Let me summarize some of the results.
First, one important invariant of a nilpotent orbit is the sizes of the Jordan blocks, which in turn is a partition. In types A, B, C, the partition turns out to be a complete invariant, while in type D, it's nearly a complete invariant.

A) In type A_n, the nilpotent orbits are in bijection with partitions of $n+1$.
B) In type B_n, the nilpotent orbits are in bijection with the partitions of $2n+1$ where even parts occur with even multiplicity.
C) In type C_n, the nilpotent orbits are in bijection with partitions of $2n$ where the odd parts occur with even multiplicity.
D) in type D_n, the nilpotent orbits are in bijection with partitions of $2n$ where the even parts occur with even multiplicity, except for "very even partitions," which are partitions consisting only of even numbersm and each very even partition corresponds to two nilpotent orbits.

We can then say what partitions the zero, minimal, subregular, and regular orbits correspond to.

Dynkin diagram	zero	minimal	subregular	regular
A_n	$[1^{n+1}]$	$[2, 1^{n-1}]$	$[n, 1]$	$[n+1]$
B_n	$[1^{2n+1}]$	$[2^2, 1^{2n-3}]$	$[2n-1, 1^2$	$[2n+1]$
C_n	$[a^{2n}]$	$[2, 1^{2n-2}]$	$[2n-2, 2]$	$[2n]$
D_n	$[1^{2n}]$	$[2^2, 1^{2n-4}]$	$[2n-3, 3]$	$[2n-1, 1]$

Example 4.17: To each regular nilpotent $x \in \mathcal{N}^{reg}$, there is a unique Borel subalgebra containing x, so that the Springer fiber $\mathcal{B}_x$ consists of a single point.

Corollary 4.18: The Springer resolution $\mu : \widetilde{\mathcal{N}} \to \mathcal{N}$ is an isomorphism on the dense open orbit $\mathcal{N}^{reg}$, justifying the name resolution (of singularities).

Example 4.19: Let $\mathfrak{g} = \mathfrak{sl}_2$. The nilpotent cone $\mathcal{N}$ was described explicitly in Example 2.5. There are two SL_2-orbits: the closed orbit consisting of just $\{0\}$, and the open dense orbit of regular nilpotent elements, which is everything else. The Springer fiber over 0 is all of $\mathcal{B} \cong \mathsf{SL}_2/\mathsf{B} \cong \mathbb{P}^1$, while the Springer fiber over any other nilpotent element is a single point.

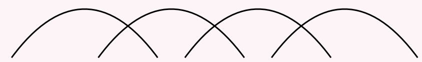
4.4 Subregular Springer fibers

Recall that the largest G-orbits in $\mathcal{N}$ always go as follows: there is a unique open dense orbit of regular nilpotent elements, there is no orbit of codimension 1, and there is a unique orbit of codimension 2, called the **subregular (nilpotent) orbit**.

The Springer fiber of an element in the subregular orbit, called a **subregular Springer fiber**, is very interesting.

The McKay correspondence gives a bijection between finite subgroups of $\mathsf{SL}_2(\mathbb{C})$ up to isomorphism, and simply-laced Dynkin diagrams (we have to associative the trivial group to the empty Dynkin graph). To a finite subgroup $H \subset \mathsf{SL}_2(\mathbb{C})$, we consider the *Kleinian singularity* $\mathbb{C}^2/H$; this has exactly one singular point, the origin. Now it's known that there's a minimal resolution $Y \to \mathbb{C}^2/H$, and this minimal resolution gives us a derived equivalence $D^b(\mathsf{Coh}(Y)) \simeq D^b(\mathsf{Coh}_H(\mathbb{A}^2))$. Furthermore, the exceptional fiber of the singular point (the origin) under this minimal resolution is known to be a union of $\mathbb{P}^1$s in a very specific format - each $\mathbb{P}^1$ is associated to a vertex in the Dynkin diagram associated to H under the McKay correspondence, and the two $\mathbb{P}^1$s intersect (at a single point) exactly when the corresponding vertices are connected by an edge. In other words, the exceptional fiber is the dual graph of the Dynkin diagram!

Example 4.20: For the Dynkin diagram A_n, corresponding to a cyclic group in $\mathrm{SL}_2(\mathbb{C})$, then the exceptional fiber consists of n $\mathbb{P}^1$s in a row, and each $\mathbb{P}^1$ intersects only its immediate left and right neighbors. So it looks something like:



.

The point is that **the subregular Springer fibers look like the exceptional fiber of the minimal resolution above**. If $\mathfrak{g}$ is simply laced (i.e., ADE), then the subregular Springer fiber is a union of $\mathbb{P}^1$s whose dual graph is the Dynkin diagram. When $\mathfrak{g}$ is not simply laced, the story is slightly different. In this case, when $\mathfrak{g}$ is of type B_n, C_n, F_4, and G_2, then the subregular Springer fiber is again a union of $\mathbb{P}^1$s whose dual graph is the Dynkin diagram A_{2n-1}, D_{n+1}, E_6, and D_4, respectively.

There's more to the story than simply coincidence: all of this plays out concretely inside the nilpotent cone. To a subregular nilpotent e in the subregular nilpotent orbit, we can construct a transverse slice S_e (for example, using the Slodowy slice) consisting of only regular elements except for e; this slice must necessarily be a dimension 2 surface. It turns out that S_e is a Kleinian singularity, i.e., one of the singular surfaces $\mathbb{C}^2/H$ for H a finite subgroup in $\mathrm{SL}_2(\mathbb{C})$, as described above, with the unique singular point (the origin) being $e \in S_e$. Then the preimage of S_e in $\widetilde{\mathcal{N}}$ under μ is exactly the minimal resolution, and therefore the Springer fiber $\mathcal{B}_e$ is exactly the exceptional fiber of the minimal resolution.

4.5 Example: Springer fibers in SL_3

Actually I just want to describe the Springer fibers of nilpotent elements in SL_3, so sorry if you were interested in the other fibers. To do so, we first need to know the SL_3-orbits in $\mathcal{N}$, which has dimension $2 \dim \mathfrak{n} = 2 \cdot 3 = 6$. We know that there is a unique dense open orbit of nilpotent regular elements, a unique codimension 2 orbit called the subregular orbit, a unique closed orbit consisting just of 0, and a unique smallest orbit larger than the 0 orbit - the minimal orbit.

By analyzing Jordan normal form, we see that **there are only three nilpotent conjugacy classes**:

a) the zero matrix $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

b) the subregular matrix $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

c) and the regular matrix $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

This is because the minimal orbit *is* the subregular orbit; for example, in Remark 4.3, we see that the partitions corresponding to the minimal and subregular orbits of A_2 are both $[2, 1]$, corresponding to the Jordan blocks of size 2 and 1. Similarly, the zero matrix has Jordan blocks of size 1, hence corresponds to the partition $[1, 1, 1]$, while the regular matrix written above is a Jordan block of size 3, hence corresponds to the partition $[3]$.

Example 4.21 (zero orbit): This one is easy. The Springer fiber over the zero element is all of $\mathcal{B}$, since every Borel subalgebra contains 0.

Example 4.22 (subregular orbit): We know from §4.4 that the Springer fiber over a subregular nilpotent element should be two $\mathbb{P}^1$s intersecting at a single point, i.e., forming the dual graph to the Dynkin diagram A_2 of SL_3. Let's see this concretely. Let's take our original representative

$$e = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Write the standard coordinates to be x_1, x_2, x_3. Then $\mathcal{B}_e$ corresponds to flags stabilized by e, namely $0 \subsetneq V_1 \subsetneq V_2 \subsetneq V_3 = \mathbb{C}^3$. Since e sends $x_1 \mapsto 0$, $x_2 \mapsto x_1$, and $x_3 \mapsto 0$, we have $\ker(e) = \langle x_1, x_3 \rangle$ and $\mathrm{im}(e) = \langle x_1 \rangle$. So we have one family of flags $0 \subset V_1 \subset \ker(e) \subset \mathbb{C}^3$; this consists of all one-dimensional subspaces of $\ker(e) \cong \mathbb{C}^2$, hence is $\mathbb{P}^1$. We have another family $0 \subset \mathrm{im}(e) \subset V_2 \subset \mathbb{C}^3$, which is all two-dimensional subspaces of $\mathbb{C}^3$ containing $\mathrm{im}(e) = \langle x_1 \rangle$; this is also $\mathbb{P}^1$. We can see that they intersect at exactly one flag, namely $0 \subset \langle x_1 \rangle \subset \langle x_1, x_3 \rangle \subset \mathbb{C}^3$. So indeed, $\mathcal{B}_e$ consists of two $\mathbb{P}^1$s intersecting at exactly one point.

Example 4.23: Let $e = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. We know that e is regular nilpotent, hence by Example 4.3, there is a unique Borel containing e, so $\mathcal{B}_e$ is a single point. In this case, it's just the Borel subalgebra $\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$. For an arbitrary regular nilpotent element e', we can obtain e' from e by conjugating by some $M \in \mathrm{SL}_3$, then the unique Borel subalgebra containing e' is just the Borel written above but conjugated by M.

4.6 Steinberg variety and the geometry of Springer fibers

One key tool in understanding the geometry and represenation theory of Springer fibers is the Steinberg variety.

Definition 4.24 (Steinberg variety): The **Steinberg variety** is defined to be

$$Z := \widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}} = \{(x, \mathfrak{b}, \mathfrak{b}') \in \mathcal{N} \times \mathcal{B} \times \mathcal{B} \mid x \in \mathfrak{b} \cap \mathfrak{b}' \cap \mathcal{N}\}.$$

Using the notation of §2.4, let Y_w denote the (diagonal) G-orbit in $\mathcal{B} \times \mathcal{B}$ corresponding to $w \in \mathbb{W}$. It turns out that the geometry of Z is fairly concrete:

Proposition 4.25 (geometry of Z): Set-theoretically, Z is a disjoint union of the conormal bundles to G-orbits in $\mathcal{B} \times \mathcal{B}$:

$$Z = \bigsqcup_{w \in \mathbb{W}} T^*_{Y_w}(\mathcal{B} \times \mathcal{B}).$$

The conormal bundles are not closed in Z, thus cannot be irreducible components. However, the closures of the conormal bundles *are* closed, and it turns out that the irreducible components of Z are exactly $\overline{T^*_{Y_w}(\mathcal{B} \times \mathcal{B})}$.

There's another way to describe the irreducible components of Z, which is by nilpotent orbits. Set $\mathbb{O} \subset \mathcal{N}$ to denote a nilpotent G-orbit, and take $Z_{\mathbb{O}} \subset Z$ to be preimage in Z:

$$\begin{tikzcd} Z_{\mathbb{O}} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, "(x,\mathfrak{b},\mathfrak{b}') \mapsto x"] \\ \mathbb{O} \arrow[r, hook] & \mathcal{N}. \end{tikzcd}$$

Proposition 4.26: For any orbit $\mathbb{O}$, $Z_{\mathbb{O}}$ is pure dimensional of dimension $2 \dim \mathfrak{n} = 2 \dim \mathcal{B}$.

We therefore have *another* way to parametrize irreducible components of Z: the irreducible components of $Z_{\mathbb{O}}$, as $\mathbb{O}$ runs over all nilpotent orbits in $\mathcal{N}$.

Now fix an orbit $\mathbb{O}$ and a point $x \in \mathbb{O}$. Let G_x denote the stabilizer of x under the adjoint action of G, and let $\mathcal{B}_x$ be its Springer fiber. To study $Z_{\mathbb{O}}$, we need to introduce

$$\widetilde{\mathcal{N}} \supset \widetilde{\mathbb{O}} := \mu^{-1}(\mathbb{O}) = \mathrm{G} \times^{\mathrm{B}} (\mathbb{O} \cap \mathfrak{n}) = \mathrm{G} \times^{\mathrm{G}_x} \mathcal{B}_x.$$

This gives $\widetilde{\mathbb{O}}$ the structure of a G-equivariant bundle over $\mathbb{O} \simeq \mathrm{G}/\mathrm{G}_x$, by the projection map $\mathrm{G} \times^{\mathrm{G}_x} \mathcal{B}_x \longrightarrow \mathrm{G}/\mathrm{G}_x$. Therefore,

$$Z_{\mathbb{O}} = \widetilde{\mathbb{O}} \times_{\mathbb{O}} \widetilde{\mathbb{O}} = \mathrm{G} \times^{\mathrm{G}_x} (\mathcal{B}_x \times \mathcal{B}_x).$$

This description is very powerful and we immediately read off several geometric consequences:

Corollary 4.27:

a) The irreducible components of $Z_{\mathbb{O}}$ are $\mathrm{G} \times^{\mathrm{G}_x} (\mathcal{B}_x^{\alpha} \times \mathcal{B}_x^{\beta})$, where $\mathcal{B}_x^{\alpha}, \mathcal{B}_x^{\beta}$ are irreducible components of $\mathcal{B}_x$.

b) The Springer fiber $\mathcal{B}_x$ is pure dimensional, of dimension $\dim \mathcal{B}_x = \dim \mathcal{B} - \frac{1}{2} \dim \mathbb{O}$.

c) By Zariski's main theorem, $\mathcal{B}_x$ is also connected.

Remark 4.28: We can actually explicitly enumerate the irreducible components of $Z_{\mathbb{O}}$. We have an action of G_x on $\mathcal{B}_x$ by conjugation, hence inducing an action on the set of irreducible components of $\mathcal{B}_x$. The connected component of G_x acts trivially on permuting the irreducible components, hence the action factors through $\pi_0(\mathrm{G}_x)$, the component group of G_x; this is a finite group. Therefore the irreducible components of $Z_{\mathbb{O}}$ are in bijection with diagonal $\pi_0(\mathrm{G}_x)$-orbits on pairs of irreducible components of $\mathcal{B}_x$.

5 Representations of the Weyl group

Our goal is to study representations of the Weyl group $\mathbb{W}$. This is a finite group, and we're not actually interested in representations of finite groups - that's a purely algebraic problem, so why are we studying this? The point is that Weyl groups are not just any finite groups - they are finite groups deeply linked to the geometry of Lie groups, and correspondingly, their representations can also be constructed geometrically. It turns out that we can recover all of their representations from Springer fibers.

5.1 Steinberg variety encodes the Weyl group

Definition 5.1: To a pure-dimensional variety X, let $H(X)$ denote the top Borel-Moore homology of X (by default, use rational coefficients $\mathbb{Q}$). It has a basis by the irreducible components of X, and since $X \circ X = X$, we have $(H_{\bullet}^{BM}(X, \mathbb{Q}), \star)$ is an associative algebra under convolution, which by dimension properties descends to an associative algebra structure on $(H(X), \star)$.

The key fact that will allow us to turn algebra into geometry is the following:

Theorem 5.2: There's an isomorphism of associative $\mathbb{Q}$-algebras $H(Z, \mathbb{Q}) \cong \mathbb{Q}[\mathbb{W}]$; the multiplication on the left side is convolution, and the multiplication on the right side is from group multiplication.

Sketch of proof. See [CG97, Theorem 3.4.1] for full details.
Broadly speaking, we consider the graph $\mathrm{Graph}(w)$ of the element $w \in \mathbb{W}$ as a subvariety in $\widetilde{\mathfrak{g}}^{sr} \times \widetilde{\mathfrak{g}}^{sr}$. Since convolution of graphs in Borel-Moore homology is just composition of functions, the classes of these subvarieties act as $w \in \mathbb{Q}[\mathbb{W}]$.

Slightly more rigorously, for each $w \in \mathbb{W}$, we take our base space to be $\mathfrak{H}_w := \mathrm{Graph}(\mathfrak{H} \xrightarrow{w} \mathfrak{H})$. We have a locally trivially fibration

$$\nu \times \nu : \widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}} \longrightarrow \mathfrak{H} \times \mathfrak{H},$$

where ν is from (4.1), and take $\widetilde{\mathfrak{g}}_w := (\nu \times \nu)^{-1}(\mathfrak{H}_w)$ to be the smooth total space. Then we have a locally trivial fibration $\nu \times \nu : \widetilde{\mathfrak{g}}_w \longrightarrow \mathfrak{H}_w$ whose special fiber at $0 \in \mathfrak{H}_w$ is $\nu^{-1}(0) \times \nu^{-1}(0) = \widetilde{\mathcal{N}} \times \widetilde{\mathcal{N}}$.

So ideally, we define some closed subvarieties $\Lambda_w \subset \widetilde{\mathfrak{g}}_w$ which play the role of Graph(w); the way to do this is to define Λ_w to be the preimage of the diagonal $\mathfrak{g}_\Delta \subset \mathfrak{g}\times\mathfrak{g}$ under the map $\mu\times\mu : \widetilde{\mathfrak{g}}_w = \widetilde{\mathfrak{g}}\times_{\mathfrak{H}_w}\widetilde{\mathfrak{g}} \to \mathfrak{g}\times\mathfrak{g}$. On one hand, restricting to the generic semisimple regular elements, we'll find that $\Lambda_w^{reg} = \mathrm{Graph}(\widetilde{\mathfrak{g}}^{sr} \xrightarrow{w} \widetilde{\mathfrak{g}}^{sr})$. On the other extreme, specializing to the special point $0 \in \mathfrak{H}$, then $\Lambda_w \cap (\nu^{-1}(0)\times\nu^{-1}(0)) \subset (\widetilde{\mathfrak{g}}\times_{\mathfrak{g}}\widetilde{\mathfrak{g}}) \cap (\widetilde{\mathcal{N}}\times\widetilde{\mathcal{N}}) = \widetilde{\mathcal{N}}\times_{\mathcal{N}}\widetilde{\mathcal{N}} = Z$.

The last step is attempting to specialize from $\mathfrak{H}_w^{reg} \to 0$. This would involve a specialization map $\lim_{h\to 0} : H(\Lambda_w^{reg},\mathbb{Q}) \to H(Z,\mathbb{Q})$, and we can define $[\Lambda_w^0]$ to be the image of the fundamental class $[\Lambda_w^{reg}]$. The $[\Lambda_w^{reg}]$ convolve with each other in the same way that elements of $\mathbb{W}$ multiply, because they're literally graphs of the $\mathbb{W}$-action. Since convolution commutes with specialization, the resulting classes $[\Lambda_w^0] \in H(Z)$ would then satisfy the same relations: $[\Lambda_w^0] \star [\Lambda_y^0] = [\Lambda_{wy}^0] \in H(Z)$. I'll mention a minor technical issue (which doesn't really detract from the motivation behind the proof): to define specialization, we need the projection $\Lambda_w \to \mathfrak{H}_w$ to be locally trivial away from 0. This is not the case: $\mathfrak{H}_w \setminus \{0\} \supsetneq \mathfrak{H}_w^{reg}$, so there's a technical step where we replace $\mathfrak{H}$ with a smaller subset for which this is true.

Finally, we just need to know that specialization map is well-defined, i.e., independent of our "parameter" h, and that these classes span $H(Z,\mathbb{Q})$, thus forming a basis. This turns out to be true, thus giving us an isomorphism $\mathbb{Q}[\mathbb{W}] \xrightarrow{\sim} H(Z,\mathbb{Q})$, sending $\mathbb{W} \ni w \mapsto [\Lambda_w^0]$. □

Corollary 5.3: $H(Z)$ is semisimple (as an algebra).

Remark 5.4: This is quite a startling fact because $\mathbb{W}$ does not act on Springer fibers $\mathcal{B}_e$ as automorphisms of varieties. Nevertheless, the $H(Z)$-action on $H(\mathcal{B}_e)$ allows us to construct a natural $\mathbb{W}$-action on $H(\mathbb{B}_e)$.

5.2 Springer correspondence

Theorem 5.1 gives us a LOT to work with. Remember that Borel-Moore homology actions come from "composition": for varieties X, Y, Z such that $X \circ Y = Z$, we have maps $H_\bullet^{BM}(X) \otimes H_\bullet^{BM}(Y) \to H_\bullet^{BM}(Z)$. If $X \circ X = X$, then $H_\bullet^{BM}(X)$ is an associative algebra, and if $X \circ Y = Y$, then $H_\bullet^{BM}(Y)$ is an $H_\bullet^{BM}(X)$-module.

Our main application will be:

Corollary 5.5: For $x \in \mathcal{N}$, we have $Z \circ \mathcal{B}_x = \mathcal{B}_x = \mathcal{B}_x \circ Z$. This means that $H(\mathcal{B}_x)$ is both a left and right $H(Z)$-module.

So we have one action from $H(Z)$; we also know that $\mathcal{B}$ carries a G-action, sending Springer fibers $g : \mathcal{B}_x \mapsto \mathcal{B}_{gxg^{-1}}$. We have to know how these two actions intertwine.

Lemma 5.6: The G-action and (either left or right; pick one) $H(Z)$-action on $H(\mathcal{B}_x)$ commute: for all $g \in \mathrm{G}$ and $z \in H(Z)$, we have

$$\begin{tikzcd} H(\mathcal{B}_x) \arrow[r, "g"] \arrow[d, "z\star-"'] & H(\mathcal{B}_{gxg^{-1}}) \arrow[d, "z\star-"] \\ H(\mathcal{B}_x) \arrow[r, "g"'] & H(\mathcal{B}_{gxg^{-1}}). \end{tikzcd}$$

Proof. For $c \in H(\mathcal{B}_x)$, we want to show that $z \star g(c) = g(z \star c)$. Since $\mathrm{G} \curvearrowright Z$, we have a map $\mathrm{G} \to \mathrm{Aut}(Z)$, therefore we already know that $g(z) \star g(c) = g(z \star c)$. But since G is assumed to be connected, it acts trivially on any homology, in particular $g|_{H(Z)} = \mathrm{id}$ and so $g(z) = z$, giving the result. □

Similarly, the stabilizer of x is G_x, so we get an action $\mathrm{G}_x \curvearrowright \mathrm{B}_x$. The action on *homology* $H(\mathcal{B}_X)$ factors through the component group $\pi_0(\mathrm{G}_x)$ (due to connected groups acting trivially on homology). Therefore:

Corollary 5.7: The (left or right) $H(Z)$-action and $\pi_0(\mathrm{G}_x)$-action on $H(\mathcal{B}_x)$ commute.

This is what really allows us to break open $H(\mathcal{B}_x)$. Let's just consider everything as left modules for simplicity. Then treating $H(\mathcal{B}_x)$ purely as a $\pi_0(\mathrm{G}_x)$-representation for now, representation theory of finite groups tells us (note that $\pi_0(\mathrm{G}_x)$ is a finite group and $H(\mathcal{B}_x)$ is finite-dimensional) that

$$H(\mathcal{B}_x, \mathbb{C}) \simeq \bigoplus_{\text{irreps } \psi \text{ of } \pi_0(\mathrm{G}_x} \psi \otimes H(\mathcal{B}_x, \mathbb{C})_\psi,$$

where $H(\mathcal{B}_x, \mathbb{C})_\psi$ are the multiplicity spaces. The commuting actions instantly tells us that **the multiplicity spaces are $H(Z)$-modules**.

> **Remark 5.8:** We use $\mathbb{C}$-coefficients to guarantee semisimplicity, but it's actually enough to use any field where all irreps of $\pi_0(\mathrm{G}_x)$ are defined. It turns out that $\mathbb{Q}$ is already sufficient for everything except E_8.

The main theorem tells us how much this strategy succeeds (hint: it's very successful). Identifying $H(Z, \mathbb{C}) \simeq \mathbb{C}[\mathbb{W}]$, then $H(Z)$-modules are just $\mathbb{W}$-representations. It turns out **all irreducible $\mathbb{W}$-representations are realized as multiplicity spaces in the Borel-Moore homology of Springer fibers**.

> **Theorem 5.9 (Springer correspondence):**
> *a)* $H(\mathcal{B}_x, \mathbb{C})_\psi$ is a simple $H(Z, \mathbb{C})$-module for all $x \in \mathcal{N}$ and all irreps ψ of $\pi_0(\mathrm{G}_x)$.
> *b)* Two multiplicity spaces $H(\mathcal{B}_x, \mathbb{C})_\psi \cong H(\mathcal{B}_y)_\gamma$ are isomorphic as $H(Z, \mathbb{C})$-modules iff (x, ψ) and (y, γ) are G-conjugate, i.e., there exists $g \in \mathrm{G}$ such that $y = gxg^{-1}$, and γ gets identified with ψ via $\mathrm{G}_y = \mathrm{G}_{gxg^{-1}} = g\mathrm{G}_x g^{-1}$.
> *c)* **All** simple $H(Z, \mathbb{C})$-modules arise in this way.
>
> In other words,
> $$\left\{H(\mathcal{B}_x, \mathbb{C})_\psi \mid \text{G-conjugacy classes of pairs } (x, \psi)\right\}$$
> forms a complete set of simple $H(Z, \mathbb{C})$-modules, equivalently, irreducible $\mathbb{W}$-representations.

> **Example 5.10:** If $e \in \mathcal{N}^{reg}$ is a regular nilpotent element, then its Springer fiber $\mathcal{B}_e$ is just a point. Then $H(\mathcal{B}_e, \mathbb{Q}) = H_0^{BM}(\mathrm{pt}, \mathbb{Q}) = \mathbb{Q}$. Since $H(Z, \mathbb{Q})$ already acts trivially on $H(\mathcal{B}_e, \mathbb{Q})$, it must act trivially on all multiplicity spaces (although there's only one anyway, for the trivial $\pi_0(\mathrm{G}_e)$-representation), hence the Springer fiber over regular nilpotent elements gives us the trivial $\mathbb{W}$-representation.

For a slightly more involved example:

> **Example 5.11 (Representations coming from the subregular orbit):** Let $e \in \mathfrak{g}$ be a subregular nilpotent element. Recall that the Springer fiber $\mathcal{B}_e$ will be a union of $\mathbb{P}^1$s, arranged in the dual graph of a corresponding Dynkin diagram Γ' (not necessarily the one giving rise to $\mathfrak{g}$). Then $\mathcal{B}_e$ is one-dimensional as a complex algebraic variety, hence its top Borel-Moore homology is $H(\mathcal{B}_e) = H_2^{BM}(\mathcal{B}_e) = H^2(\mathcal{B}_e)$, and we can identify the $\mathbb{P}^1$s with the vertices of the Dynkin diagram Γ'. Then the action of $\pi_0(\mathrm{G}_e) \simeq \mathrm{Aut}(\Gamma')$, and the action of $\pi_0(\mathrm{G}_e)$ on $H(\mathcal{B}_e)$ is exactly the action of $\mathrm{Aut}(\Gamma')$ permuting the vertices of Γ'.
>
> Taking the trivial representation ψ of $\pi_0(\mathrm{G}_e)$, the corresponding multiplicity space in $H(\mathcal{B}_e)$ is exactly the $\pi_0(\mathrm{G}_e)$-invariants: $H(\mathcal{B}_e)_\psi = H(\mathcal{B}_e)^{\pi_0(\mathrm{G}_e)}$. Now essentially by how the Dynkin diagram Γ' is constructed from $\mathfrak{g}$, it turns out that the $\mathrm{Aut}(\Gamma')$-invariants of the vector space with basis given by vertices of Γ' has dimension equal to the rank of $\mathfrak{g}$. In fact, the $\mathbb{W}$-module structure on $H(\mathcal{B}_e)^{\pi_0(\mathrm{G}_e)}$ is exactly the reflection representation of $\mathbb{W}$ on $\mathfrak{H}^*$. Thus under the Springer correspondence, the pair of subregular nilpotent and trivial representation (e, ψ) corresponds to the reflection representation of $\mathbb{W}$.

Example 5.12 (zero orbit): The Springer fiber over 0 is all of $\mathcal{B}$, so we have an action of $\mathbb{W}$ on $H_\bullet^{BM}(\mathcal{B}) = H^\bullet(\mathcal{B})$ (recall that $\mathcal{B}$ is smooth and compact, so Borel-Moore homology is just singular cohomology). What $\mathbb{W}$-representation does this correspond to under the Springer correspondence? We have an abstract action of $\mathbb{W}$ on $H_\bullet^{BM}(\mathcal{B})$ via convolution from $H(Z)$, but let's see it more concretely. (Note that this time we are looking at the entire homology space, not just the top one.)

First, it's important to note that **there is no natural action of $\mathbb{W}$ on $\mathcal{B}$**. But there's a relatively straightforward way to construct an action of *some* manifestation of the Weyl group W_{T}, by choosing $\mathsf{T} \subset \mathsf{G}$ and realizing $W_{\mathsf{T}} = N_{\mathsf{G}}(\mathsf{T})/\mathsf{T}$. Note that G/T is an affine space bundle over $\mathsf{G}/\mathsf{B} \cong \mathcal{B}$, so their singular cohomologies and Borel-Moore homologies match. But G/T is easier to work with because $W_{\mathsf{T}} = N_{\mathsf{G}}(\mathsf{T})/\mathsf{T}$ naturally acts on G/T on the right, hence we get an induced action of $W_{\mathsf{T}} \curvearrowright H^\bullet(\mathsf{G}/T) = H^\bullet(\mathcal{B})$.

To ensure it extends to a well-defined action of $\mathbb{W}$, we need to know that every choice of (T, B) gives compatible actions. In other words, any two pairs are G-conjugate by some $g \in \mathsf{G}$, and the corresponding manifestations of their Weyl groups W_{T} are also conjugate by the same exact g; furthermore, the isomorphisms $\mathsf{G}/\mathsf{B}' \simeq \mathcal{B} \simeq \mathsf{G}/\mathsf{B}''$ are conjugate *again* by the same $g \in \mathsf{G}$. So all of these actions are compatible, allowing us to have a well-defined action of abstract $\mathbb{W}$ on $H_\bullet^{BM}(\mathcal{B})$.

The main result is that these two $\mathbb{W}$-actions on $H_\bullet^{BM}(\mathcal{B})$ coincide, and furthermore, $H_\bullet^{BM}(\mathcal{B})$ is the regular representation of $\mathbb{W}$. If we restrict to the top space $H(\mathcal{B}) \subset H_\bullet^{BM}(\mathcal{B})$, then it's the sign representation of $\mathbb{W}$.

Remark 5.13: Let me point out that in certain sources, such as [Yun16], the Springer correspondence is done using singular cohomology of the Springer fibers, $H^\bullet(\mathcal{B}_e)$. This is not actually that different from our presentation: since $\mathcal{B}_e$ is compact (μ is a proper map, hence fibers are compact), Borel-Moore homology agrees with singular homology, so $H_\bullet^{BM}(\mathcal{B}_e) = H_\bullet(\mathcal{B}_e)$. On the other hand, the universal coefficients theorem implies that (over a field) singular cohomology is the dual space of singular homology, so as a $\mathbb{W}$-module, $H^\bullet(\mathcal{B}_e)$ is the dual representation of $H_\bullet(\mathcal{B}_e) = H_\bullet^{BM}(\mathcal{B}_e)$, and the top singular cohomology group is the dual representation of the top Borel-Moore homology group. Since $\mathbb{W}$ is a finite group, the dual of a finite-dimensional irreducible representation is again irreducible, so the Springer correspondence using singular cohomology will just be the dual of our version.

5.3 Example: $\mathfrak{sl}_n$

Let's look at the case $\mathfrak{sl}_n$ concretely. The advantage here is that we can make all $\pi_0(\mathsf{G}_x)$ trivial, so that $H(\mathcal{B}_x)$ themselves are exactly the $\mathbb{W} = S_n$-irreps.

First, a word for how to trivialize the component groups. All constructions that we did - for example, flag variety, Springer fibers, etc. - relied only on the semisimple Lie algebra $\mathfrak{sl}_n$. But all of these constructions are the same when we take a reductive $\mathfrak{g}$ whose semisimple component $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{sl}_n$. So we replace $\mathfrak{sl}_n$ with $\mathfrak{gl}_n$, and SL_n with GL_n. We have the same nilpotent cone, nilpotent conjugacy classes, flag variety, and Springer fibers. The advantage is that:

Lemma 5.14: For all $x \in \mathfrak{gl}_n$, we have that $\pi_0((\mathsf{GL}_n)_x) = 1$.

Proof. The proof is basically by blindly identifying a bunch of stuff. First, the stabilizer group of x is

$$(\mathsf{GL}_n)_x = \{y \in \mathsf{GL}_n \mid xy = yx\} \subset \{y \in \mathrm{Mat}_n = \mathfrak{gl}_n \mid xy = yx\} = \mathbb{C}^r,$$

the latter of which is a vector space. Inside this vector space, $(\mathsf{GL}_n)_x$ is the complement of a (complex) codimension 1 hypersurface (namely, the hypersurface cut out by the determinant). But this is real codimension 2, hence $\pi_0((\mathsf{GL}_n)_x) = 1$. $\square$

Remark 5.15: It's *not* true that $\pi_0((\mathrm{SL}_n)_x) = 1$ for every $x \in \mathfrak{sl}_n$. So upgrading from semisimple to reductive increased our Lie group enough to ensure that the stabilizer groups are connected, thus allowing us to say that the multiplicity spaces are the same as the entirety of $H(\mathcal{B}_x)$.

What are the implications? Well, this lemma allows us to say that the multiplicity spaces are the same as the entirety of $H(\mathcal{B}_x)$, since the component groups are now trivial, and the only irreducible representation of the trivial group is trivial. In other words,

$$H(\mathcal{B}_x, \mathbb{Q}) = \mathbb{Q} \otimes_{\mathbb{Q}} H(\mathcal{B}_x, \mathbb{Q})_{\text{trivial}} = H(\mathcal{B}_x, \mathbb{Q})_{\text{trivial}}.$$

Therefore, the Springer correspondence (5.2) becomes:

Theorem 5.16 (Springer correspondence for $\mathfrak{sl}_n$):

a) $H(\mathcal{B}_x)$ is a simple $H(Z) = \mathbb{Q}[S_n]$-module for all $x \in \mathcal{N}$.

b) $H(\mathcal{B}_x) \cong H(\mathcal{B}_y)$ as $H(Z) = \mathbb{Q}[S_n]$-modules iff x, y are SL_n-conjugate, i.e., lie in the same nilpotent orbit.

c) All simple $H(Z) \simeq \mathbb{Q}[S_n]$-modules arise in this way.

As a result, we have a bijection:

$$\begin{tikzcd}
\{\text{irreducible } S_n\text{-representations}\} \arrow[r, leftrightarrow, "\text{Springer fibers}"] & \{\mathrm{SL}_n\text{-orbits in } \mathcal{N}\} \arrow[r, equal] \arrow[d, leftrightarrow] & \{\text{conjugacy classes of } n \times n \text{ nilpotent matrices}\} \\
& \{\text{partitions of } n\} \arrow[ul, "\text{Young tableux}"] \arrow[ur, "\text{Jordan blocks}"'] &
\end{tikzcd}$$

Example 5.17: Let $\mathfrak{g} = \mathfrak{sl}_3$. Let's see what the Springer correspondence says concretely. There are three partitions of 3, namely $[3]$, $[2, 1]$, and $[1^3]$, corresponding to the three nilpotent orbits: regular, subregular, and zero orbit.

a) The regular orbit corresponds to the partition $[3]$, and corresponds to the trivial S_3-represenation.

b) The subregular orbit corresponds to the partition $[2, 1]$, and corresponds to the standard (two-dimensional) representation of S_3.

c) The zero orbit corresponds to the partition $[1^3]$, and corresponds to the sign representation of S_3.

5.4 And beyond

These methods of studying representations from geometric origins are extremely fruitful. As one example, we can even construct all representations of $\mathfrak{sl}_n$ in a similar manner, using partial flag varieties instead: see [CG97, §4].

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