Sato-Kashiwara filtration

Merrick Cai

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1 Introduction

This short note aims to explain the Sato-Kashiwara filtration on $\mathcal{D}$-modules. It will primarily aim to cover [1, §5].

Suppose we have a $\mathbb{C}$-scheme X, and let's say it's affine for the sake of simplicity. Then the sheaf $\mathbb{D}\mathcal{D}_X$ is a quasicoherent sheaf of (noncommutative) algebras over X. Since X is affine in this case, $\mathcal{D}_X$ is just the quasicoherent sheaf associated to a noncommutative algebra which by abuse of notation we also write as $\mathcal{D}_X$. Recall the construction:

Definition 1.1.

Let X be a scheme. To any affine open $U \subset X$, we define the algebra $\mathcal{D}_U$ as a subalgebra of $\mathcal{E}nd(\mathcal{O}_X)$ generated by $\mathcal{O}_X$ and $\Theta_X = \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_X)$, the sheaf of vector fields.

Example 1.2.

Let $X = \mathbb{A}^n$. We have $\mathcal{O}_X = \mathbb{C}[x_1, \dots, x_n]$ while $\Theta_X = \mathbb{C}[\partial_{x_1}, \dots, \partial_{x_n}]$ where $\partial_{x_i} = \frac{\partial}{\partial x_i}$ is the partial differential operator. Then

$$\mathcal{D}_{\mathbb{A}^n} = \mathbb{C}[x_1, \dots, x_n, \partial_{x_1}, \dots, \partial_{x_n}]$$

subject to the usual relations between x_i and ∂_{x_j}.

$\mathcal{D}$-modules come with an important filtration known as the **order filtration**, and since they are quasicoherent sheaves of infinite-dimensional algebras, these filtrations are important to get a handle on their modules.

Definition 1.3.

The **order filtration** on $\mathcal{D}_U$ for U a smooth affine scheme is constructed as follows. We define $F_0\mathcal{D}_U \coloneqq \mathcal{O}_U$. Then we iteratively define

$$F_i\mathcal{D}_U \coloneqq \{f \in \mathcal{D}_U \mid [f, \mathcal{O}_U] \subset F_{i-1}\mathcal{D}_U\}.$$

When $U = \operatorname{Spec} \mathbb{A}^n$, we have another filtration as well.

Definition 1.4.

The **Bernstein filtration** on $\mathcal{D}_{\mathbb{A}^n}$ is defined by

$$F_i\mathcal{D}_{\mathbb{A}^n} = \bigoplus_{|\alpha|+|\beta| \le i} \mathbb{C} \cdot \{x^\alpha \partial^\beta\},$$

where α, β are multi-indices in $\mathbb{Z}^n_{\ge 0}$. (In other words, take the generators $x_1, \dots, x_n, \partial_{x_1}, \dots, \partial_{x_n}$ and allow for all formal polynomials of degree $\le i$ in these generators.)

Example 1.5.

On $X = \mathbb{A}^n$, the order filtration gives

$$F_i\mathcal{D}_{\mathbb{A}^n} = \mathbb{C}[x_1, \dots, x_n] \cdot \{\partial^I \mid |I| \le i\},$$

where $I \in \mathbb{Z}^n_{\ge 0}$ is an n-tuple indexing the monomial in the partial differential operators, i.e.

$$\partial^{(a_1, \dots, a_n)} \coloneqq \partial^{a_1}_{x_1} \cdots \partial^{a_n}_{x_n}.$$

To any $\mathcal{D}$-module M (i.e. a module over $\mathcal{D}_X$) we want it to have a filtration compatible with M, along with certain other finiteness conditions which give the filtration good properties; these are called **good filtrations**. These good filtrations are important for us to work with the associated graded modules, but they always exist and any two good filtrations are actually not very far off from each other. However, there are more filtrations that we want to consider which study other characteristics - in this note, we take a look at the Sato-Kashiwara filtration, which measures the "dimension" and "codimension" of $\mathcal{D}$-modules.

2 Generalities

Let's work in a more general setting. However, if at any point you wish to revert to a more concrete example, please see Example 2.3.

2.1 Setup

Notation 2.1.

We always work over a fixed algebraically closed field of characteristic zero, e.g. $\mathbb{C}$ (I'll write $\mathbb{C}$ everywhere but you can replace it with another field as you wish). Let R be a filtered ring with filtration $F_\bullet R$ and assume that we have the following properties:

- $F_0 R$ is Noetherian;
- $S_\bullet = \operatorname{gr}^F R$ is commutative and generated (over $F_0 R$) by finitely many elements of degree 1;
- $S = S_\bullet$ is a regular ring with all maximal ideals of codimension d. This is equivalent to requiring that $\operatorname{Spec} S$ is pure of dimension d (and this assumption will allow us to define pure in the codimension sense, see Definition 3.6).

Additionally, **all R-modules in the remainder of this article are assumed to be finitely-generated**.

The assumptions imply the following facts:

Proposition 2.2.

- R is both left and right Noetherian.
- F is a filtration on both R and R^{op}, hence $\operatorname{gr}^F R$ can be identified with $\operatorname{gr}^F R^{\mathrm{op}}$.
- The global dimension of S is equal to d. (This is the Auslander-Buchsbaum-Serre theorem.)

Example 2.3.

Our main example for R is the algebra of differential operators $\mathcal{D}_A$ on some affine n-dimensional variety $\operatorname{Spec} A$, with the order filtration. In this case $d = 2n$. Therefore, if at any point you wish to make this note more concrete, I invite you to replace all R with $\mathcal{D}_A$ (or even $\mathcal{D}_{\mathbb{A}^n}$, and S with $\mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]$).

Alternatively, if $A = \mathbb{C}[x_1, \dots, x_n]$ and $\mathcal{D}_{\mathbb{A}^n}$ is given the Bernstein filtration, then we also get $d = 2n$.

2.2 Filtrations

Since R comes with a filtration $F_\bullet R$, we want every R-module M to come with a filtration too. The basic idea is that this filtration on M should be compatible with the filtration on R, plus satisfy a few finiteness conditions.

Definition 2.4.

A **filtration** on M compatible with R is a filtration $F_\bullet M$ such that:

- $F_i M \subseteq F_{i+1} M$;
- $F_q M = 0$ for $q \ll 0$;
- $M = \bigcup_{q \in \mathbb{Z}} F_q M$;
- $F_p R \cdot F_q M \subseteq F_{p+q} M$ for all $p, q \in \mathbb{Z}$.

The filtration is called a **good filtration** if it also satisfies:

- $F_q M$ is a finitely-generated (equivalently, finitely-present, i.e. coherent) $F_0 R$-module for every q;
- There exists some $t \in \mathbb{Z}$ such that $F_{p+t} M = F_p R \cdot F_t M$ for all $p \geq 0$ (this is called *generated at level* q).

Theorem 2.5.

A good filtration always exists on M (so long as M is finitely-generated).

Proof. Since M is finitely-generated, just take the finite collection of generators and form the $F_0 R$-submodule M_0 generated by these elements. Then declare $F_p M = F_p R \cdot M_0$ for every p. This is a good filtration. □

Proposition 2.6.

Let M and N be R-modules, and choose good filtrations on them. For every i, we have a noncanonical filtration on the abelian groups $\operatorname{Ext}^i_R(M, N)$ such that:

(1) The associated graded $\operatorname{gr}\operatorname{Ext}^i_R(M,N)$ is isomorphic to a subquotient of $\operatorname{Ext}^i_S(\operatorname{gr} M, \operatorname{gr} N)$.

(2) If for all $\ell < i$ that $\operatorname{Ext}^\ell_S(\operatorname{gr} M, \operatorname{gr} N) = 0$, then $\operatorname{gr}(\operatorname{Ext}^i_R(M,N))$ is isomorphic to a subgroup of $\operatorname{Ext}^i_S(\operatorname{gr} M, \operatorname{gr} N)$.

Proof. The basic idea for obtaining these filtrations is to choose filtered free resolutions of M, which are very concrete (even if they are also very noncanonical), and then we can study their convergence using spectral sequences. The proof is not important for us, so for more details see [1, Proposition 5.2]. □

Remark 2.7.

In Proposition 2.6, we obtain noncanonical filtrations on $\operatorname{Ext}^i_R(M,N)$ as abelian groups. When $N = R$, then each $\operatorname{Ext}^i_R(M,R)$ has the natural structure of a (left) R^{op}-module, and the filtration constructed in the proposition is even a good filtration with respect to the filtration $F_\bullet R^{\mathrm{op}}$ on R^{op}.

2.3 Dualizing functors

Definition 2.8.

Let $D^b_{\text{f.g.}}(R)$ be the bounded derived category of the abelian category $R\text{−}\mathsf{mod}^{\text{f.g.}}$ of finitely-generated R-modules.

Every element in $D^b_{\text{f.g.}}(R)$ can be represented by a finite complex of finitely-generated R-modules.

Now recall that for rings A, B, then $(A-B)$-bimodules give functors between the categories $A\text{−}\mathsf{mod}$ and $B^{\mathrm{op}}\text{−}\mathsf{mod}$ (by tensor and Hom). Deriving these functors gives eaxct functors between the derived categories. We apply that in this case: R is an $(R-R)$-bimodule.

Definition 2.9.

We have an exact functor

$$\begin{aligned} \mathbf{D}_R : D^b_{\text{f.g.}}(R) &\to D^b_{\text{f.g.}}(R^{\mathrm{op}}), \\ M &\mapsto \mathbf{R}\mathrm{Hom}_R(M,R). \end{aligned}$$

Theorem 2.10.

The contravariant functor $\mathbf{D}_R$ is an anti-equivalence of categories, with inverse $\mathbf{D}_{R^{\mathrm{op}}}$.

3 Dimension and codimension of module

In the case of $\mathcal{D}$-modules, we can use the existence of good filtrations to define the *characteristic variety* of a $\mathcal{D}$-module M. It turns out that the characteristic variety is actually independent of the choice of good filtration on M. We then define the dimension of a $\mathcal{D}$-module M to be the dimension of its characteristic variety.

In this subsection we'll do exactly the same thing in our more general setting of R. Recall that **we assume all R-modules to be finitely-generated**.

3.1 Dimension

Definition 3.1.

Let M be an R-module, and choose a good filtration. Then we define the **characteristic variety** $\operatorname{Char}(M)$ of M to be the support of $\operatorname{gr} M$ in $\operatorname{Spec} S$. In other words, it is the closed subset of $\operatorname{Spec} S$ defined by the ideal $\sqrt{\operatorname{Ann}_S(\operatorname{gr} M)}$, the radical of the annihilator ideal of $\operatorname{gr} M$ in S.

We define the **dimension** of M to be $\dim \operatorname{Char}(M)$.

For an object $u \in D^b_{\text{f.g.}}(R)$, define

$$\operatorname{Char}(u) := \bigcup_{i \in \mathbb{Z}} \operatorname{Char}(H^i(u)).$$

It turns out that $\operatorname{Char}(M)$ is independent of the choice of good filtration, which is crucial (and allows us to make a well-defined definition of dimension). Also note that $\dim(M) \le \dim \operatorname{Spec} S = d$.

Remark 3.2.

If we have a short exact sequence of R-modules

$$0 \to M_1 \to M_2 \to M_3 \to 0,$$

then

$$\operatorname{Char}(M_2) = \operatorname{Char}(M_1) \cup \operatorname{Char}(M_3) \implies \dim(M_2) = \max(\dim(M_1), \dim(M_3)).$$

This readily extends to exact triangles $u \to v \to w \to u[1]$ in $D^b_{\text{f.g.}}(R)$: we have $\operatorname{Char}(v) \subseteq \operatorname{Char}(u) \cup \operatorname{Char}(w)$.

The important fact is that **characteristic varieties are preserved by duality**. This is a generalization of the fact that characteristic varieties of $\mathcal{D}_X$-modules are involutive with respect to the symplectic structure of T^*X.

Proposition 3.3.

For any $u \in D^b_{\text{f.g.}}(R)$, we have $\text{Char}(u) = \text{Char}(\mathbf{D}_R(u))$.

3.2 Codimension

We also have a notion of *codimension* of an R-module, which fortunately turns out to be $d - \dim(M)$.

Definition 3.4.

The **codimension** of an R-module M is defined to be

$$j(M) := \min_{i \in \mathbb{Z}_{\geq 0}} \left(H^i(\mathbf{D}_R(M)) = \text{Ext}^i_R(M, R) \neq 0 \right).$$

In other words, it is **the smallest integer i for which the ith cohomology of $\mathbf{D}_R(M)$ does not vanish**.

Codimension can be thought of as "order of smallness" for a module: typically, we think of higher cohomology groups as "infinitesimal structure" of a module/complex. This matches the intuition with higher codimension being lower dimension, hence "smaller."

Note that codimension is always well-defined for nonzero M, since $\mathbf{D}_R$ is an equivalence of (triangulated) categories and hence takes M to something nonzero in $D^b_{\text{f.g.}}(R^{\text{op}})$, which implies that some cohomology group must be nonzero (i.e., $\mathbf{D}_R(M)$ is not quasi-isomorphic to the zero complex).

Remark 3.5.

If R is a commutative Noetherian ring, then $j(M) = \text{depth}(\text{Ann}_R(M), R)$. If R is also assumed to be regular (hence Cohen-Macaulay) with all maximal ideals of the same codimension, then $j(M) = \text{codim}(\text{Ann}_R(M)) = \dim(R) - \dim(M)$.

Definition 3.6.

M is **pure** if every nonzero submodule has the same codimension, i.e. for every $0 \neq N \subseteq M$, we have $j(N) = j(M)$.

This definition is linked directly to the notion "pure dimension" and should be thought of as such: see Proposition 3.8. A module is pure iff the characteristic variety is pure.

The Examples 4.2 and 4.3 showcase two pure modules.

Theorem 3.7.

Let M be an R-module.

(1) $j(M) = d - \dim(M)$.

(2) If $H^i(\mathbf{D}_R(M)) \neq 0$, then $j(H^i(\mathbf{D}_R(M))) \geq i$.

(3) $H^{j(M)}(\mathbf{D}_R(M))$ is a pure R-module whose codimension is equal to $j(M)$, the codimension of M.

This useful theorem tells us 1) that codimension lives up to its name, and 2) the codimensions of the R^{op}-modules $H^i(\mathbf{D}_R(M))$, from which codimension of M is defined upon.

Proposition 3.8.

- M is pure of codimension ℓ iff $\mathrm{Char}(M)$ is pure of dimension $d-\ell$ (iff there is a good filtration on M such that $\mathrm{gr}\, M$ is pure of codimension ℓ).
- Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of nonzero R-modules. Then $j(M_1) = \min(j(M_1), j(M_2))$.
- If $0 \neq M_1, M_2 \subset M$ then $j(M_1 + M_2) = \min(j(M_1), j(M_2))$.

Remark 3.9.

One important consequence of (1) in Theorem 3.7 is that dimension of a module does not depend on the filtration on the ring R, so long as the associated graded of R has the same dimension. This is because the codimension j is defined independently of any filtration. Therefore, so long as we pick a filtration F' (different from our original filtration F) for which $\dim \mathrm{gr}^{F'} R = d = \dim \mathrm{gr}^F R$, then dimension of a module is still the same! (However, the characteristic varieties are not, since we can't even say that $\mathrm{Spec}\, \mathrm{gr}^{F'} R \cong \mathrm{Spec}\, \mathrm{gr}^F R$.)

In particular, for M a module over $\mathcal{D}_{\mathbb{A}^n}$, the notion of dimension via the order filtration and the Bernstein filtration coincide.

4 Sato-Kashiwara filtration

The goal in this section is to study the Sato-Kashiwara filtration, which is a decreasing filtration on an R-module by certain R-submodules.

4.1 Gabber filtration

Definition 4.1.

Define the **Gabber filtration** on M to be

$$G^i(M) := \sum_{\substack{N \subseteq M, \\ j(N) \geq i}} N = \sum_{\substack{N \subseteq M, \\ \dim(N) \leq d-i}} N.$$

If $M \neq 0$, we have that

$$M = G^{j(M)}(M) \supseteq G^{j(M)+1}(M) \supseteq \cdots \supseteq G^d(M) \supseteq G^{d+1}(M) = 0.$$

Furthermore, if $G^i(M) \neq 0$, then it follows from the third statement of Proposition 3.8 that $j(G^i(M)) \geq i$, hence $G^i(M)$ is characterized by being the unique largest submodule of M of codimension $\geq i$.

The submodule $G^i(M)$ is not always exactly codimension i: this only happens when $G^{i+1}(M) = 0$; if $G^{i+1}(M) \neq 0$, then the quotient $G^i(M)/G^{i+1}(M)$ is pure of codimension i (or is 0). As a corollary, $M \neq 0$ is pure iff $G^{j(M)+1}(M) = 0$.

We can think of the Gabber filtration as being the algebraic equivalent to the filtration on the characteristic variety by its subvarieties of increasing codimension (respectively, decreasing dimension). To each i, then $G^i(M)$ will be the largest submodule supported on a subvariety of M of codimension-i (which is codimension $i - j(M)$ in $\mathrm{Char}(M)$). The characteristic variety $\mathrm{Char}(M)$ may not be pure of dimension $d - j(M)$, and if it is not, the irreducible components of higher codimension $\ell > j(M)$ (i.e., lower dimension) will precisely correspond to changes in the Gabber filtration at G^ℓ from $G^{\ell-1}$.

In particular, the obstruction to M being pure is the non-triviality of the Gabber filtration (i.e. there are terms which are neither M nor 0 but something in-between), and so these "higher-order" terms (where higher order is "smaller") correspond to the existence of irreducible components of that codimension. If $\mathrm{Char}(M)$ were pure, each irreducible component would be pure of codimension j, equivalently pure of dimension $d - j$ (since one of the assumptions is that $\operatorname{Spec} S$ is pure of dimension d). In turn, any irreducible component of lower dimension, equivalently higher codimension, will be detected by a nontrivial (i.e., proper nonzero) submodule appearing in the Gabber filtration. So the Gabber filtration contains the obstructions to pureness. In the case of $\mathcal{D}_X$-modules, holonomic is equivalent to pure of codimension $d/2 = \dim X$, hence the Gabber filtration contains the obstruction to being holonomic as well.

Example 4.2.

Take $R = \mathcal{D}_{\mathbb{A}^n}$ with the order filtration and $M = \mathcal{O}_{\mathbb{A}^n}$ with the trivial filtration, i.e. $F_i M = M$ for all $i \geq 0$. Then $\operatorname{gr} R \cong \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$

where $\deg x_i = 0$ and $\deg y_i = 1$, while $(\operatorname{gr} M)_0 = \mathbb{C}[x_1, \ldots, x_n]$ and $(\operatorname{gr} M)_{>0} = 0$. It follows that y_i act by 0 and x_i act by the standard action, so $\operatorname{Char}(M) = \mathbb{A}^n \subset \mathbb{A}^{2n}$, which is a pure variety of dimension n (hence pure of codimension n). The absence of irreducible subschemes of $\mathbb{A}^n$ with dimension less than n implies that the higher Gabber filtration terms vanish. Then it's clear that $G^{>n}(M) = 0$ and $G^{\leq n}(M) = M$ (this also implies $\mathcal{O}_{\mathbb{A}^n}$ is holonomic).

Example 4.3.

Take $R = M = \mathcal{D}_{\mathbb{A}^n}$, both with order filtration. Then $\operatorname{gr} R = \operatorname{gr} M =\cong \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ with $\deg x_i = 0$ and $\deg y_i = 1$, with the obvious action. The characteristic variety is the entirety of $\operatorname{Spec} \operatorname{gr} R \cong \mathbb{A}^{2n}$, hence is pure of dimension $d = 2n$. This corresponds to the Gabber filtration being zero after $j = d - d = 0$. It follows that $G^0(M) = M$ and $G^{>0}(M) = 0$.

This is a **very concrete definition** but it is a bit difficult to work with since there are no obvious functorial properties. As a result, we introduce the alternative, more abstract, characterization.

4.2 Sato-Kashiwara filtration

Since we're working in the derived category $D^b_{\text{f.g.}}(R)$, we can use the **truncation functors** $\tau_{\geq i}$. We have a natural map $X \to \tau_{\geq i} X$ for every object $X \in D^b_{\text{f.g.}}(R^{\text{op}})$ (in fact, there's even an exact triangle $\tau_{\leq i-1} X \to X \to \tau_{\geq i} X$ which is functorial and characterizes the truncation functors), inducing isomorphisms on cohomology groups $H^{\geq i}$ and zero maps on cohomology groups $H^{<i}$. Then applying $\mathbf{D}_{R^{\text{op}}}$, we obtain a natural map $\mathbf{D}_{R^{\text{op}}} \tau_{\geq i} X \to \mathbf{D}_{R^{\text{op}}}$ for every $X \in D^b_{\text{f.g.}}(R)^{\text{op}}$.

Definition 4.4.

For any R-module M, we define the ith **Sato-Kashiwara filtration** of M to be

$$\begin{aligned} S^i(M) &\coloneqq \operatorname{im}\left[H^0\left(\mathbf{D}_{R^{\text{op}}} \tau_{\geq i} \mathbf{D}_R(M)\right) \to H^0\left(\mathbf{D}_{R^{\text{op}}} \mathbf{D}_R(M)\right)\right], \\ &\cong \operatorname{im}\left[H^0\left(\mathbf{D}_{R^{\text{op}}} \tau_{\geq i} \mathbf{D}_R(M)\right) \to M\right]. \end{aligned}$$

First, we need to make sense of why this is a filtration.

Proposition 4.5.

The $S^i(M)$ form a finite, functorial, decreasing filtration of M.

Proof. The reason why these form a filtration is simple: there's a natural map $\tau_{\geq i}\mathbf{D}_R(M) \to \tau_{\geq i+1}\mathbf{D}_R(M)$, and this map behaves well under the functorial operations we perform to produce $S^i(M)$.

To see that it's finite, we just have to note that $S^{<0}(M) = 0$ and $S^{>d}(M) = M$ since the global dimension of R is at most d (the global dimension of S), and $\mathbf{D}_R$ is computed using projective resolutions (which therefore can be made finite of length $\leq d$. It follows that the truncation functors $\tau_{>d}$ kill the entire complex $\tau_{>d}\mathbf{D}_R(M) = 0$, hence $S^{>d}(M) = 0$.

Functoriality is clear, since the $S^i(M)$ are defined functorially. As a result, if $f : M_1 \to M_2$, then $f(S^i(M_1)) \subseteq S^i(M_2)$. ☐

Remark 4.6.

In fact, the maps $H^0(\mathbf{D}_{R^{\mathrm{op}}}\tau_{\geq i}\mathbf{D}_R(M)) \to M$ defining $S^i(M)$ are injective. This can be seen by the long exact sequence associated to $\mathbf{D}_{R^{\mathrm{op}}}$ applied to the exact triangle $\tau_{\leq i-1}X \to X \to \tau_{\geq i}X$ for $X = \mathbf{D}_R(M)$, and then carefully analyzing the spectral sequence to see that the $H^{-1}(\mathbf{D}_{R^{\mathrm{op}}}\tau_{\leq i-1}\mathbf{D}_R(M)) = 0$.

4.3 The main theorem

We now have two interesting filtrations: the Gabber filtration and the Sato-Kashiwara filtration. The Gabber filtration is very concrete and so is easier to compute, but appears difficult to prove stuff, since it's defined concretely rather than abstractly. In Example 4.2 and Example 4.3 we computed the Gabber filtration without too much difficulty. The Sato-Kashiwara filtration is a bit difficult to understand concretely due to its abstract definition, but is easy to work with (abstractly) since it's defined functorially. We can see right from the definition that it's not exactly easy to compute with: it's defined in terms of $\mathbf{R}$Hom, which means we need to take a finite projective resolution and apply Hom to it, then apply truncation functors and dualize again, which is just a pain.

The main theorem (which should be a huge relief) is that **these two filtrations coincide**, so we get the best of both worlds: it's easy to compute from the Gabber perspective, and use to work with abstractly from the Sato-Kashiwara perspective.

Theorem 4.7.

The Gabber filtration and the Sato-Kashiwara filtration coincide:

$$G^i(M) = S^i(M) \quad \text{for all } i \in \mathbb{Z}.$$

This might come as a surprise, but if we look closer, the definition of the Sato-Kashiwara filtration actually *encodes* the Gabber filtration. The codimension measures the non-vanishing of the cohomology of $\mathbf{D}_R M$. Therefore to get the subset of codimension $\geq i$, we need to kill the cohomology groups $H^{<i}$. How do we do that? Well, the natural method is to take the truncation $\tau_{\geq i}\mathbf{D}_R(M)$: whichever submodule "gives" this under $\mathbf{D}_R$ will, by definition, have codimension $\geq i$. So naturally we use $\mathbf{D}_R^{-1} = \mathbf{D}_{R^{\mathrm{op}}}$ to go back to the world of R-modules, resulting in $\mathbf{D}_{R^{\mathrm{op}}}\tau_{\geq i}\mathbf{D}_R(M)$. The only issue is that this isn't actually a submodule of M... nor even a module. To finally rectify that problem, we just apply H^0 and this *does* give us the submodule we want.

Proof. The first inclusion we need is $S^i(M) \subseteq G^i(M)$. Recall that one of the defining characteristics of $G^i(M)$ is that it is the unique largest submodule of M which has codimension $\geq i$. Therefore it suffices to show that $j(S^i(M)) \geq i$, as then it will automatically be a submodule of $G^i(M)$.

First, the exact triangle $H^i(\mathbf{D}_R(M))[-i] \to \tau_{\geq i}\mathbf{D}_R(M) \to \tau_{\geq i+1}\mathbf{D}_R(M)$ gives us the long exact sequence

$$0 \to H^0(\mathbf{D}_{R^{\mathrm{op}}}\tau_{\geq i+1}\mathbf{D}_R(M)) \to H^0(\mathbf{D}_{R^{\mathrm{op}}}\tau_{\geq i}\mathbf{D}_R(M)) \to H^0(\mathbf{D}_{R^{\mathrm{op}}}(H^i(\mathbf{D}_R(M)[-i]))) \to \dots.$$

Now Remark 4.6 implies that we can actually identify the H^0 with S^{i+1} and S^i themselves. On the other hand,

$$\begin{aligned} H^0(\mathbf{D}_{R^{\mathrm{op}}}(H^i(\mathbf{D}_R(M)[-i]))) &\cong \mathrm{Ext}^0_{R^{\mathrm{op}}}(H^i(\mathbf{D}_R(M))[-i], R^{\mathrm{op}}), \\ &\cong \mathrm{Ext}^i_{R^{\mathrm{op}}}(H^i(\mathbf{D}_R(M)), R^{\mathrm{op}}), \\ &\cong H^i(\mathbf{D}_{R^{\mathrm{op}}}(H^i(\mathbf{D}_R(M)))), \end{aligned}$$

so we can apply (2) of Theorem 3.7 to see that the codimension of this term is $\geq i$ (assuming it's nonzero). As a result,

$$0 \to S^{i+1}(M) \to S^i(M) \to \text{something of codimension } \geq i \to \dots.$$

From this, we can begin with $i = d$ (using the fact that codimension is bounded above by d) and apply descending induction to obtain that $S^i(M)$ is sandwiched between two things of codimension $\geq i+1$ and $\geq i$ (respectively), hence $S^i(M)$ also has codimension $\geq i$, so $S^i(M) \subseteq G^i(M)$.

Now it remains to prove the reverse inclusion, that $G^i(M) \subseteq S^i(M)$. Naturally if $G^i(M) = 0$ then $S^i(M) \subseteq G^i(M) = 0$ must also be 0. So let's assume $G^i(M) \neq 0$. Then by definition the codimension of $G^i(M) \geq i$, so by definition the map $\mathbf{D}_R(G^i) \to \tau_{\geq i}\mathbf{D}_R(G^i)$ is a quasi-isomorphism. Applying exact functor $\mathbf{D}_{R^{\mathrm{op}}}$, we find the quasi-isomorphism

$$\mathbf{D}_{R^{\mathrm{op}}}\tau_{\geq i}\mathbf{D}_R(G^i) \xrightarrow{\text{qis}} \mathbf{D}_{R^{\mathrm{op}}}\mathbf{D}_R(G^i) \cong G^i,$$

hence on the level of H^0 we find that

$$S^i(G^i) \xrightarrow{\sim} G^i$$

either by identifying $H^0(\mathbf{D}_{R^{\mathrm{op}}}\tau_{\geq i}\mathbf{D}_R(G^i)) \hookrightarrow H^0(M)$ with its image $S^i(G^i)$, or by considering the built-in maps to $\mathbf{D}_R(M)$ commutaive with the quasi-isomorphism $\mathbf{D}_R(G^i) \to \tau_{\geq i}\mathbf{D}_R(G^i)$. (The first method is simpler, but the second method has the advantage of making it abundantly clear that that the isomorphism $S^i(G^i(M)) \cong G^i$ is actually an **equality** of submodules of M.)

By functoriality of S^i (see Proposition 4.5) we find that the inclusion $G^i(M) \hookrightarrow M$ gives us the inclusion $G^i = S^i(G^i) \hookrightarrow S^i$, hence $G^i \subseteq S^i$. This gives us the reverse inclusion we need, and we conclude that $G^i(M) = S^i(M)$. □

References

[1] Mircea Mustata, *D-modules and singularities lecture notes for math 732, winter 2023.*