$\mathcal{D}$-modules

Merrick Cai

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1 Introduction

Let X be a smooth irreducible $\mathbb{C}$-variety. Loosely speaking, $\mathcal{D}$-modules are just modules over a sheaf of differential operators. Many times we want not just an action of the regular functions $\mathcal{O}_X$, but also the tangent sheaf $\mathcal{T}_X$. For example, the tangent bundle always has a natural action on any vector bundle over X (from differential geometry).

This leads us naturally to the notion of a $\mathcal{D}_X$-module, which is a module for the action of $\mathcal{O}_X$ and $\mathcal{T}_X$ simultaneously.

$\mathcal{D}$-modules turn out to be extremely important in a number of settings, but particularly in geometric representation theory. For example, representations of Lie algebras $\mathfrak{g}$ can be understood via $\mathcal{D}$-modules on the flag variety, via the Beilinson-Bernstein localization theorem. Another interesting setting is holonomic $\mathcal{D}$-modules, where we have a so-called six-functor formalism, which is a particularly nice setting to do math due to an abundance of very nice properties and interactions between these functors.

1.1 Purpose

This note is my own recollection and summary of the most important basic facts about $\mathcal{D}$-modules. They are heavily based on [Mus], which I used as an alternative to the more well-known [HTT07].

1.2 Conventions

We'll always work over $\mathbb{C}$, but you may replace this with any algebraically closed field of characteristic 0. Also, generally speaking, all of our schemes are smooth irreducible algebraic $\mathbb{C}$-varieties, although this will be clarified in each result.

1.3 Notation

Let me say it right now. The f_+ functor will be the star-pushforward f_* in the six-functor formalism. Likewise, the $f^\dagger$ functor will be the shriek-pullback $f^!$ in the six-functor formalism. So if you're more comfortable with that, you may replace each of the f_+ and $f^\dagger$ with the six-functor formalism.

2 Construction and basic properties

We'll first define $\mathcal{D}_X$, the main object of interest, and explain some basic properties. Fix X to be a smooth irreducible $\mathbb{C}$-variety. In algebraic geometry, we're often concerned about $\mathcal{O}_X$-modules, that is, sheaves of abelian groups equipped with an action by regular functions on $\mathcal{O}_X$. But oftentimes, for example in differential geometry, there's also a natural action by tangent vectors, so we should consider modules not just over $\mathcal{O}_X$, but also $\mathcal{T}_X$, the tangent sheaf. The goal of $\mathcal{D}_X$ is to package both of them together.

2.1 The fast definition

The basic idea is just that: we want a sheaf of algebras which not only includes regular functions, but also tangent vectors. The tangent vectors will be interpreted as derivations of $\mathcal{O}_X$.

Definition 2.1 ($\mathcal{D}_X$): Both $\mathcal{O}_X$ and $\mathcal{T}_X$ are sheaves of $\mathbb{C}$-linear endomorphisms of $\mathcal{O}_X$; $\mathcal{O}_X$ acts on $\mathcal{O}_X$ by left multiplication, while $\mathcal{T}_X$ acts on $\mathcal{O}_X$ by derivations.
Let $\mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)$ be the sheaf of $\mathbb{C}$-linear endomorphisms of $\mathcal{O}_X$. We define the **sheaf of differential operators** $\mathcal{D}_X$ to be the subsheaf of $\mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)$ generated (as a sheaf of $\mathbb{C}$-algebras) by (the images of) $\mathcal{O}_X$ and $\mathcal{T}_X$.

Example 2.2 ($\mathcal{D}_{\mathbb{A}^n}$): Let $X = \mathbb{A}^n$. Then $\mathcal{O}_{\mathbb{A}^n}$ corresponds to the ring $\mathbb{C}[x_1, \dots, x_n]$ while $\mathcal{T}_{\mathbb{A}^n}$ is the sheaf of tangent vectors, which have a basis given by $\partial_{x_1}, \dots, \partial_{x_n}$, suggestively named: they act by derivations on $\mathbb{C}[x_1, \dots, x_n]$ via the partial differential operators as in their notation.
It follows that

$$\mathcal{D}_{\mathbb{A}^n} = \mathbb{C}[x_1, \dots, x_n, \partial_{x_1}, \dots, \partial_{x_n}],$$

the algebra of polynomials and partial differential operators. This is also known as the **Weyl algebra**.

Note: it's not actually a polynomial algebra in $2n$ variables, since $[\partial_{x_i}, x_i] = 1$ as an endomorphism of $\mathbb{C}[x_1, x_2, \dots, x_n]$, so this is sort of an abuse of notation. We could write it instead as

$$\mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]/(y_i x_i - x_i y_i - 1).$$

2.2 Alternative definition: Grothendieck differential operators

There's another way to define $\mathcal{D}_X$ inductively, which is known as Grothendieck differential operators.

Definition 2.3 (Grothendieck differential operators, order filtration): For nonnegative integers $n \geq 0$, we inductively define the sheaf $F_n\mathcal{D}_X$ of **Grothendieck differential operators (of order $\leq p$)** on X as follows:

a) $F_0\mathcal{D}_X = \mathcal{O}_X$ as a subsheaf of $\mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)$.

b) For $n \geq 1$, $F_n\mathcal{D}_X \subset \mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)$ is the subsheaf of those $f \in \mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)$ for which $[f, \mathcal{O}_X] \subseteq F_{n-1}\mathcal{D}_X$.

We then define

$$\mathcal{D}_X = \bigcup_{n\geq 0} F_n\mathcal{D}_X \subset \mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)$$

to be the union over all $F_n\mathcal{D}_X$. Thus $\mathcal{D}_X$ has a filtration given by the $F_n\mathcal{D}_X$; it's easy to verify that $\mathcal{D}_X$ is a sheaf of algebras, that $F_n\mathcal{D}_X$ is indeed a filtration making $\mathcal{D}_X$ a sheaf of *filtered* algebras, and that all $F_n\mathcal{D}_X$ and $\mathcal{D}_X$ itself are subsheaves of $\mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)$. The filtration $F_n\mathcal{D}_X$ is called the **order filtration** on $\mathcal{D}_X$.

Example 2.4 ($F_1\mathcal{D}_X$): Let's compute the first filtration for $\mathcal{D}_X$. Working affine-locally, let $R = \mathcal{O}_X(U)$, so $F_1\mathcal{D}_X(U) = R$.

Now for $F_1\mathcal{D}_X(U)$, we need endomorphisms of R whose commutator with R lands in R. In other words, for $P \in F_1\mathcal{D}_X(U)$ and $f \in R$, we have $[P, f] = u \in R$. In particular, $[P, f](1) = u \cdot 1 = u \implies P(f) - f \cdot P(1) = u$. In other words, for each $g \in R$ (treated as a module),

$$[Pf - fP](g) = ug \implies P(fg) - f \cdot P(g) = g \cdot (P(f) - f \cdot P(1)) \implies P(fg) = fP(g) + gP(f) - gP(1),$$

so $Q := P - P(1)$ is a derivation. It follows that $F_1\mathcal{D}_X(U) = R + \mathrm{Der}_{\mathbb{C}}(R) \subset \mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)(U) = \mathrm{End}_{\mathbb{C}}(R)$.

Example 2.5 (Order filtration for $\mathcal{D}_{\mathbb{A}^n}$): Let $X = \mathbb{A}^n$, so that $\mathcal{O}_{\mathbb{A}^n}$ corresponds to the ring $\mathbb{C}[x_1, \ldots, x_n]$. Then

$$F_k\mathcal{D}_{\mathbb{A}^n} = \bigoplus_{|\alpha|\leq k} \mathbb{C}[x_1, \ldots, x_n] \cdot \partial^\alpha,$$

where $\alpha \in \mathbb{Z}^n_{\geq 0}$ is some n-tuple $(\alpha_1, \ldots, \alpha_n)$ and $\partial^\alpha := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$. It follows that

$$\mathcal{D}_{\mathbb{A}^n} = \bigoplus_{\alpha\in\mathbb{Z}^n_{\geq 0}} \mathbb{C}[x_1, \ldots, x_n] \cdot \partial^\alpha.$$

Lemma 2.6: For each m, n we have

$$F_m\mathcal{D}_X, F_n\mathcal{D}_X] \subseteq F_{m+n-1}\mathcal{D}_X.$$

In particular, the associated graded $\mathrm{gr}^F_\bullet(\mathcal{D}_X)$ is a sheaf of commutative (and graded) rings, and in fact $\mathrm{gr}^F_\bullet(\mathcal{D}_X) \simeq \mathcal{S}ym^\bullet_{\mathcal{O}_X}(\mathcal{T}_X)$.

Proof. The basic idea is to induct on $m + n$ and use the Jacobi identity: for $f \in \mathcal{O}_X$, we consider that

$$[[A, B], f] = [A, [B, f]] + [[A, f], B]\,,$$

and apply the inductive hypothesis while keeping track of where each commutator lives. Specifically, $[-, f]$ lowers the degree of the filtration by one, so the right hand side ends up being part of the induction hypothesis and both terms live in $F_{m+n-2}\mathcal{D}_X$, from which we conclude using the definition of $F_{m+n-1}\mathcal{D}_X$ (as stuff which upon commutating with f, lives in $F_{m+n-2}\mathcal{D}_X$).

The explicit description is just a matter of looking affine-locally. □

Corollary 2.7: For X a smooth complex variety, and U an affine open subset of X, then $\mathcal{D}_X(U)$ is both left and right Noetherian.

Proof. It's a general fact that (left or right) Noetherian-ness can be deduced from the associated graded, so it suffices to check if the associated graded is (left or right) Noetherian. To see that part, we just need to recall that $\mathrm{gr}^F_\bullet(\mathcal{D}_X) \simeq \mathcal{S}ym^\bullet_{\mathcal{O}_X}(\mathcal{T}_X)$, which is both left and right Noetherian. □

2.3 Basic facts about $\mathcal{D}_X$

Proposition 2.8: $\mathcal{D}_X$ is a quasicoherent $\mathcal{O}_X$-module.

Note that it is *not* a coherent $\mathcal{O}_X$-module, as it is almost always infinitely generated as an $\mathcal{O}_X$-module; for example, see Example 2.5.

Knowing that $\mathcal{D}_X$ is quasicoherent, we'd like to know how it looks on each affine open. One way is to use the construction given in Definition 2.1.

Proposition 2.9: Let $X = \operatorname{Spec} R$ be a smooth affine $\mathbb{C}$-variety, with tangent sheaf $\mathcal{T}_X$ given by the R-module $T_R = \mathrm{Der}_{\mathbb{C}}(R)$. Then $\mathcal{D}_X$ is the associative algebra freely generated by $\{\widetilde{a} \mid a \in R\}$ and $\{\widetilde{D} \mid D \in T_R\}$, subject to the relations:

- $\widetilde{a_1 a_2} = \widetilde{a_1} \cdot \widetilde{a_2}$,
- $\widetilde{aD} = \widetilde{a} \cdot \widetilde{D}$,
- $[\widetilde{D}, \widetilde{a}] = \widetilde{D(a)}$,
- $[\widetilde{D_1}, \widetilde{D_2}] = \widetilde{[D_1, D_2]}$.

This is just another way to say that $\mathcal{D}_X$ is built out of $\mathcal{O}_X$ and $\mathcal{T}_X$ (affine-locally, but it implies globally since $\mathcal{D}_X$ is quasicoherent).

Now we get to what a $\mathcal{D}$-module is. It's actually all in the name.

Definition 3.1 ($\mathcal{D}$-module): A left (or right) **$\mathcal{D}$-module** on (a smooth irreducible complex variety) X is a sheaf of left (or right) $\mathcal{D}_X$-modules.

It's really quite simple! A $\mathcal{D}$-module on X is just a module for $\mathcal{D}_X$. So the study of $\mathcal{D}$-modules returns us to the familiar world of modules over rings - in this case, a specific $\mathbb{C}$-algebra.

Note that since $\mathcal{O}_X \hookrightarrow \mathcal{D}_X$, any $\mathcal{D}_X$-module is automatically an $\mathcal{O}_X$-module.

Example 3.2: In the obvious way, $\mathcal{D}_X$ carries both a left and right $\mathcal{D}_X$-module structure. Similarly, since $\mathcal{D}_X$ was defined to be a subsheaf of $\mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)$, the sheaf $\mathcal{O}_X$ has a natural left $\mathcal{D}_X$-module structure.

There are several important types of $\mathcal{D}$-modules.

Definition 3.3 (quasicoherent $\mathcal{D}$-module): A $\mathcal{D}_X$-module is **quasicoherent** if it is quasicoherent as an $\mathcal{O}_X$-module.

Definition 3.4 (coherent $\mathcal{D}_X$-module): A (left or right) $\mathcal{D}_X$-module $\mathcal{M}$ is **coherent** if it is quasicoherent, and it is locally finitely generated over $\mathcal{D}_X$, i.e., over each (or equivalently some cover consisting of) affine open U, the module $\mathcal{M}(U)$ is a finitely-generated (left or right) $\mathcal{D}_X(U)$-module.

Note that coherent $\mathcal{D}_X$-modules are often *not* coherent as $\mathcal{O}_X$-modules! For example, $\mathcal{D}_X$ is a coherent $\mathcal{D}_X$-module, but it is certainly not coherent as an $\mathcal{O}_X$-module.

It's not hard to see that the category of left and right quasicoherent and coherent $\mathcal{D}_X$-modules is **abelian**. We'll denote the category of left quasicoherent $\mathcal{D}_X$-modules by $\mathrm{QCoh}(\mathcal{D}_X)$, and the category of left coherent $\mathcal{D}_X$-modules by

$\mathrm{Coh}(\mathcal{D}_X)$. Similarly, we'll denote the corresponding categories of right $\mathcal{D}_X$-modules by $\mathrm{QCoh}(\mathcal{D}_X^{op})$ and $\mathrm{Coh}(\mathcal{D}_X^{op})$.

There is one more type of $\mathcal{D}_X$-module, which is the nicest of all. These are the $\mathcal{D}_X$-modules which are not just coherent, but coherent as $\mathcal{O}_X$-modules. In fact, these are always locally free.

Proposition 3.5: A (left) $\mathcal{D}_X$-module which is coherent as an $\mathcal{O}_X$-module is a locally free $\mathcal{O}_X$-module.

3.1 Integrable connections

The top-down approach to building $\mathcal{D}$-modules is to give a module for the sheaf of algebras $\mathcal{D}_X$. As clean as the definition is, this is actually somewhat annoying, since at this point you've probably only seen $\mathcal{O}_X$-modules in algebraic geometry and probably have never seen people actually trying to write down abelian groups with an action of $\mathcal{D}_X$. We can rectify this by instead starting with $\mathcal{O}_X$-modules and specifying some extra conditions, called *integrable connections*, which will be equivalent to giving a $\mathcal{D}_X$-module structure. In essence, we start with an $\mathcal{O}_X$-module and then specify the action of the derivations $\mathcal{T}_X$, which (assuming somne compatibility relations) will be enough to specify the action of the entire $\mathcal{D}_X$.

Once again, X will be a smooth irreducible $\mathbb{C}$-variety.

Definition 3.6 (connection): A **connection** on an $\mathcal{O}_X$-module $\mathcal{M}$ is a $\mathbb{C}$-linear map

$$\nabla : \mathcal{M} \to \Omega_X \otimes_{\mathcal{O}_X} \mathcal{M}$$

satisfying the Leibniz condition:

$$\nabla(f \cdot m) = f\nabla(m) + df \otimes m \quad \text{for all} \quad f \in \mathcal{O}_X,\ m \in \mathcal{M}.$$

For a connection ∇ on $\mathcal{M}$, we can extend it to a series of maps

$$\nabla : \Omega_X^i \otimes_{\mathcal{O}_X} \mathcal{M} \to \Omega_X^{i+1} \otimes_{\mathcal{O}_X} \mathcal{M}$$

by setting

$$\nabla(\eta \otimes m) := d\eta \otimes m + (-1)^{|\eta|}\eta \wedge \nabla(m).$$

Definition 3.7 (integrable connection): A connection ∇ on $\mathcal{M}$ is **integrable** or **flat** if $\nabla^2 = 0$ as a map $\mathcal{M} \to \Omega_X^2 \otimes_{\mathcal{O}_X} \mathcal{M}$.

Proposition 3.8: To an $\mathcal{O}_X$-module $\mathcal{M}$, giving it a left $\mathcal{D}_X$-module structure is equivalent to giving it an integrable connection.

Remark 3.9: This is perhaps better spelled out in the following way.

First, Ω_X is a locally free $\mathcal{O}_X$-module, and its dual is $\mathcal{T}_X$. So giving a connection ∇ on $\mathcal{M}$ is equivalent to giving a map $\nabla : \mathcal{D}_X \to \mathcal{E}nd_{\mathbb{C}}(\mathcal{M})$, sending $v \mapsto \nabla_v$, satisfying:

i) $\nabla_{f\xi} = f\nabla_\xi$ for all $f \in \mathcal{O}_X$ (this is the property of $\mathcal{O}_X$-linearity);

ii) $\nabla_\xi(f \cdot m) = f\nabla_\xi(m) + \xi(f)m$ for all $f \in \mathcal{O}_X$ and $m \in \mathcal{M}$ (this is the Leibniz rule).

The condition of being *integrable* is equivalent to

iii) $\nabla_{[v,w]} = [\nabla_v, \nabla_w]$.

So to give a (left) $\mathcal{D}_X$-module structure to an $\mathcal{O}_X$-module $\mathcal{M}$, it suffices to construct a map $\nabla : \mathcal{D}_X \to \mathcal{E}nd_{\mathbb{C}}(\mathcal{M})$ satisfying i, ii, and iii.

The proof is fairly straightforward; it basically amounts to the explicit description of $\mathcal{D}_X$ on affine opens given in Proposition 2.9.

3.2 Good filtrations and characteristic varieties

As we saw in Proposition 2.8, $\mathcal{D}_X$ is quasicoherent (as an $\mathcal{O}_X$-module). It's an infinitely-generated $\mathcal{O}_X$-module, which gives us much the same difficulties as studying infinite-dimensional vector spaces. Since such things are rarely naturally graded, the best we can usually hope for is some sort of *filtration*. In this case, $\mathcal{D}_X$ already comes equipped with the order filtration (see Definition 2.3), such that its associated graded is a sheaf of *commutative* $\mathbb{C}$-algebras (see Lemma 2.6). So if we could find a way to put a filtration on a $\mathcal{D}_X$-module $\mathcal{M}$, compatible with the order filtration on $\mathcal{D}_X$, then the associated graded would be a module over $\mathrm{gr}^F_\bullet(\mathcal{D}_X)$, which is just a sheaf of commutative rings, and we would be able to return to the world of commutative algebra and standard algebraic geometry. We'll do just that.

Our story starts out with associating to $\mathcal{D}_X$ a scheme, rather than just a sheaf. To do so, we need it to be commutative, so we first take the associated graded (see Lemma 2.6) to obtain

$$\mathrm{gr}^F_\bullet(\mathcal{D}_X) \simeq \mathcal{S}ym^\bullet_{\mathcal{O}_X}(\mathcal{T}_X).$$

Letting $\pi : T^*X \to X$ be the canonical projection map from the cotangent bundle, we have that

$$\mathrm{gr}^F_\bullet(\mathcal{D}_X) \simeq \pi_*(\mathcal{O}_{T^*X}) \iff \underline{\mathrm{Spec}}_X\left(\mathrm{gr}^F_\bullet(\mathcal{D}_X)\right) \simeq T^*X.$$

All of this is just a globalized version of saying that on each affine open $U \subset X$,

$$\mathrm{Spec}\,\mathrm{gr}^F_\bullet(\mathcal{D}_X)(U) \simeq T^*U.$$

(We just need to treat the sheaf of rings on U as a single ring, using the fact that affine schemes can be identified with their coordinate ring of global sections.) So the scheme we will work with is the cotangent bundle of X; we should think of T^*X as the scheme-ified version of $\mathcal{D}_X$.

Remark 3.10: The cotangent bundle on a (smooth irreducible complex) variety X has many nice properties, which includes having a natural symplectic structure on it.

Definition 3.11 (good filtration): Let $\mathcal{M}$ be a quasicoherent $\mathcal{D}_X$-module. A family $\{F_n\mathcal{M}\}_{n\in\mathbb{Z}_{\geq 0}}$ of quasicoherent $\mathcal{O}_X$-submodules of $\mathcal{M}$ is a **filtration** if:

(i) $F_n\mathcal{M} \subseteq F_{n+1}\mathcal{M}$ for all n,
(ii) $\mathcal{M} = \bigcup_{n\in\mathbb{Z}_{\geq 0}} F_n\mathcal{M}$,
(iii) $F_n\mathcal{D}_X \cdot F_m\mathcal{M} \subseteq F_{n+m}\mathcal{M}$ for all n, m.

The filtration is called a **good filtration** if it satisfies the following additional finiteness conditions:

(iv) $F_n\mathcal{M}$ is a *coherent* $\mathcal{O}_X$-module for every n,
(v) There is a q such that $F_{n+q} = F_n\mathcal{D}_X \cdot F_q\mathcal{M}$ for all $n \geq 0$; i.e., the filtration is completely generated after a finite number (q) of steps.

Example 3.12: The order filtration is a good filtration of $\mathcal{D}_X$ (viewed as a left $\mathcal{D}_X$-module): it is generated in degree 1.

First, a quick fact: any two good filtrations are "similar," or not too far apart from each other.

Lemma 3.13: If $F_\bullet\mathcal{M}$ and $F'_\bullet\mathcal{M}$ are good filtrations on $\mathcal{M}$, then there is some $\ell \geq 0$ such that $F'_{k-\ell}\mathcal{M} \subseteq F_k\mathcal{M} \subseteq F_{k+\ell}\mathcal{M}$ for all k.

Proof sketch. The idea is basically that $\mathcal{M}$ must be generated in some degree with respect to each of the good filtrations, and those two degrees pretty much determine the remainder of the filtration. Therefore we can just compare the two base degrees, and possibly enlarge to handle the finitely many cases under them. □

We have several immediate questions:

a) When do good filtrations exist and how can we construct them;

b) When is a filtration good and how can we recognize them;

c) What can we do with them?

Let's answer these one by one.

Proposition 3.14: A quasicoherent $\mathcal{D}_X$-module $\mathcal{M}$ has a good filtration iff it is a **coherent** $\mathcal{D}_X$-module (i.e., locally finitely generated over $\mathcal{D}_X$). In this case, we can construct a good filtration, as explained in the proof.

Proof. One direction is clear. If $\mathcal{M}$ has a good filtration, then by property (v) above, it will be generated at some level $F_q\mathcal{M}$, which is a coherent $\mathcal{O}_X$-module.

In the other direction, assume $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module. We can construct a good filtration as follows.

- We first construct a coherent $\mathcal{O}_X$-submodule $\mathcal{M}_0 \subseteq \mathcal{M}$ which generates $\mathcal{M}$ as a $\mathcal{D}_X$-module. This can be done as follows.
  - Cover X by finitely many affine opens.
  - On each affine open U, there's a coherent $\mathcal{O}_U$-submodule $\mathcal{M}'_U \subseteq \mathcal{M}|_U$ generating $\mathcal{M}|_U$ over $\mathcal{D}_X|_U$.
  - We can extend the $\mathcal{M}'_U$ sheaves to sheaves on all of X: for each U, we can find a coherent $\mathcal{O}_X$-submodule $\mathcal{M}_U \subseteq \mathcal{M}$ for which $\mathcal{M}_U|_U = \mathcal{M}'_U$.
  - Now just take $\mathcal{M}_0 := \sum_U \mathcal{M}_U$. This is clearly a coherent $\mathcal{O}_X$-submodule which generates $\mathcal{M}$ as a $\mathcal{D}_X$-module.
- To construct the good filtration, we can just declare it to be generated in degree 0 and define $F_n\mathcal{M} := F_n\mathcal{D}_X \cdot \mathcal{M}_0$. This is easily seen to be a good filtration.

□

Proposition 3.15: A filtration $F_\bullet\mathcal{M}$ on a quasicoherent $\mathcal{D}_X$-module $\mathcal{M}$ is good iff $\operatorname{gr}^F_\bullet(\mathcal{M})$ is a locally finitely-generated $\operatorname{gr}^F_\bullet(\mathcal{D}_X)$-module.

The proof is more or less just translating the definition of "locally finitely-generated" and condition (iv) above.

Finally, we answer the last question: what do we use these good filtrations for?

Definition 3.16 (characteristic variety): Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module on X with good filtration $F_\bullet\mathcal{M}$. Then $\operatorname{gr}^F_\bullet(\mathcal{M}) \simeq \pi_*(\mathcal{F})$ for some coherent sheaf $\mathcal{F}$ on T^*X, and $\pi : T^*X \to X$ as before.

The **characteristic variety** $\mathsf{Char}(\mathcal{M}) \subset T^*X$ is the support of $\mathcal{F}$, i.e., the closed subset of T^*X defined by the annihilator of $\mathcal{F}$ in $\mathcal{O}_{T^*X}$.

Remark 3.17: Recall that our previous description of $\operatorname{gr}^F_\bullet(\mathcal{D}_X)$ as the relative Spec is just a globalized version of saying that for each affine open $U \subset X$, that $\operatorname{Spec} \operatorname{gr}^F_\bullet(\mathcal{D}_X)(U) \simeq T^*U$. In the same way, the characteristic variety is just saying that for each affine open $U \subset X$, then $\operatorname{gr}^F_\bullet(\mathcal{M})(U)$ is a module for the commutative ring $\operatorname{gr}^F_\bullet(\mathcal{D}_X)(U)$, and $\mathsf{Char}(\mathcal{M}) \cap T^*U \subset T^*X$ gives exactly the support.

So the characteristic variety is just a globalized version of gluing the supports of the associated graded module as a module over the associated graded of $\mathcal{D}_X$ on each affine open.

There's some things we need to check.

Proposition 3.18: The characteristic variety of $\mathcal{M}$ is independent of the choice of good filtration.

Proof sketch. This pretty much amounts to using that any two good filtrations are similar, as in Lemma 3.13, to then show that the (radical of the) annihilators are the same. □

Remark 3.19:
$$\mathsf{Char}(\mathcal{M}) = \emptyset \iff \mathrm{gr}^F_\bullet(\mathcal{M}) = 0 \iff \mathcal{M} = 0.$$

Definition 3.20 (dimension): The **dimension** $\dim(\mathcal{M})$ of a coherent $\mathcal{D}_X$-module $\mathcal{M}$ is the dimension of the characteristic variety $\mathsf{Char}(\mathcal{M})$. (It is clear that $\dim(\mathcal{M}) \le \dim T^*X = 2\dim X$.)

Example 3.21 ($\dim(\mathcal{D}_X)$): We can use the order filtration on $\mathcal{D}_X$ as a good filtration for $\mathcal{D}_X$ (as a coherent left $\mathcal{D}_X$-module). Then the annihilator is just the zero ideal sheaf, which corresponds to (the closed set equal to) the entirety of T^*X, i.e. $\mathsf{Char}(\mathcal{M}) = T^*X$. It follows that $\dim(\mathcal{D}_X) = 2\dim X$.

Example 3.22 (dimension of an $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module): Say $\mathcal{M}$ is a (nonzero) $\mathcal{D}_X$-module, coherent as an $\mathcal{O}_X$-module. Then we can choose a good filtration given by $F_k\mathcal{M} = \mathcal{M}$ for all $k \ge 0$. It follows that $\mathsf{Char}(\mathcal{M}) = \pi^{-1}(0) \subset T^*X$, the zero-section. Therefore $\dim(\mathcal{M}) = \dim X$.

Example 3.23: Let $X = \mathbb{A}^1$ and $\mathcal{M} = \mathbb{C}[x, x^{-1}]/\mathbb{C}[x]$. We have the order filtration on $\mathcal{D}_{\mathbb{A}^1} = \mathbb{C}[x, \partial_x]$, identifying $\mathrm{gr}^F_\bullet(\mathcal{D}_{\mathbb{A}^1}) = \mathbb{C}[x, y]$, with y corresponding to the image of ∂_x.
Then we have a good filtration given by $F_k\mathcal{M} = \mathbb{C}\{x^{\ge -k}\}/\mathbb{C}[x]$. But x acts by 0, while y acts by nonzero so $\mathsf{Char}(\mathcal{M}) = V(x) = \operatorname{Spec}\mathbb{C}[x, y]/(x) = \pi^{-1}(0)$, the zero-fiber in $T^*\mathbb{A}^1 \simeq \mathbb{A}^2$. Therefore $\dim(\mathcal{M}) = 1$.

Let us review quickly how to compare dimensions of $\mathcal{D}$-modules in short exact sequences.

Proposition 3.24: Suppose we have a short exact sequence of coherent $\mathcal{D}_X$-modules
$$0 \to \mathcal{M}_1 \to \mathcal{M} \to \mathcal{M}_2 \to 0.$$
Suppose $\mathcal{M}$ has a good filtration. Then:

- We can induce good filtrations on $\mathcal{M}_1$ (by pulling back) and $\mathcal{M}_2$ (by taking the image). This gives us a short exact sequence of $\mathrm{gr}^F_\bullet(\mathcal{D}_X)$-modules.
- Consequently, $\mathsf{Char}(\mathcal{M}) = \mathsf{Char}(\mathcal{M}_1) \cup \mathsf{Char}(\mathcal{M}_2)$, and hence $\dim(\mathcal{M}) = \max\{\dim(\mathcal{M}_1), \dim(\mathcal{M}_2)\}$.

The proofs are straightforward.

We already have an upper bound on the dimension: the characteristic variety is a subvariety of T^*X, hence is at most $\dim T^*X = 2\dim X$. What about a lower bound? In fact, there's a fundamental result:

Theorem 3.25 (Bernstein's inequality): Let X be a smooth n-dimensional variety, and $\mathcal{M}$ a nonzero coherent $\mathcal{D}_X$-module. Then $\dim(\mathcal{M}) \ge n$, and furthermore every irreducible component of $\mathsf{Char}(\mathcal{M})$ has dimension $\ge n$.

We'll defer the proof for now, and prove it in §4.7. (The reason is that we need Kashiwara's equivalence theorem 4.21.)

3.3 Equivalence of left and right $\mathcal{D}_X$-modules

Everything we discussed so far applies equally well for right $\mathcal{D}_X$-modules, i.e., $\mathcal{D}_X^{op}$-modules. In particular, we can talk about quasicoherent and coherent $\mathcal{D}_X^{op}$-modules, good filtrations, and characteristic varieties. So how are left and right $\mathcal{D}_X$-modules related?

The answer is that they're equivalent, with a fairly simple functor to go between these two categories.

Proposition 3.26 (equivalence of left and right $\mathcal{D}_X$-modules): We have inverse equivalences of categories

$$\mathcal{D}_X\text{−mod} \underset{\mathcal{N}\otimes_{\mathcal{O}_X}\omega_X^{-1} \leftarrow\mathcal{N}}{\overset{\mathcal{M}\mapsto\omega_X\otimes_{\mathcal{O}_X}\mathcal{M}}{\rightleftarrows}} \mathcal{D}_X^{op}\text{−mod}$$

which preserve relevant properties such as quasicoherence, coherence, etc.

Proof sketch. The main idea is that

$$\mathcal{D}_X^{op} \simeq \omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X^{-1}.$$

□

This result enables us to switch between left and right $\mathcal{D}_X$-modules as we wish. This turns out to be convenient: for example, the direct image functor is more naturally written using right $\mathcal{D}_X$-modules.

Remark 3.27: After we switch between $\mathcal{D}_X$−mod and $\mathcal{D}_X^{op}$−mod, one might ask what happens to the action map of say, $\mathcal{D}_X \curvearrowright \mathcal{M} \mapsto \mathcal{M}^{op} \curvearrowleft \mathcal{D}_X^{op}$. The answer is that there is an involution $\tau : \mathcal{D}_X \xrightarrow{\sim} \mathcal{D}_X^{op}$ which is the identity on $\mathcal{O}_X$ and sends $\tau(\partial_i) = -\partial_i$ (after choosing coordinates on X). It's clear that $\tau^2 = \mathrm{Id}$. We'll also abuse notation by denoting $\tau(\mathcal{M})$ to be the opposite $\mathcal{D}$-module to $\mathcal{M}$: that is, if $\mathcal{M}$ is a left $\mathcal{D}$-module, then $\tau(\mathcal{M})$ is the corresponding right $\mathcal{D}$-module, and vice versa.

Explicitly, this tells us that if we take a right $\mathcal{D}_X$-module $\mathcal{M}$, then the action of $\varphi \in \mathcal{D}_X$ on $m \in \tau(\mathcal{M})$ is given the action of $\tau(\varphi) \in \mathcal{D}_X^{op}$ on $m \in \mathcal{M}$.

4 Functors on $\mathcal{D}$-modules

Suppose we have a map $f : X \to Y$ of smooth irreducible $\mathbb{C}$-varieties. In algebraic geometry, we have induced maps ${}^\circ f_* : \mathcal{O}_X\text{−mod} \to \mathcal{O}_Y\text{−mod}$ and ${}^\circ f^* : \mathcal{O}_Y\text{−mod} \to \mathcal{O}_X\text{−mod}$ on quasicoherent sheaves. (The $\circ$ indicates the underived functors.) We'll construct the analogues for $\mathcal{D}$-modules and discuss some consequences.

First, some notation.

Definition 4.1: We'll denote the following:

- $D^b(\mathcal{D}_X)$ for the bounded derived category of D_X−mod.
- $D^b_{qc}(\mathcal{D}_X)$ for the triangulated subcategory of $D^b(\mathcal{D}_X)$ of quasicoherent $\mathcal{D}_X$-modules (i.e., all cohomologies are quasicoherent).
- $D^b_{coh}(\mathcal{D}_X)$ for the triangulated subcategory of $D^b(\mathcal{D}_X)$ of coherent $\mathcal{D}_X$-modules (i.e., all cohomologies are coherent).
- For right $\mathcal{D}_X$-modules, replace all of the $\mathcal{D}_X$ with $\mathcal{D}_X^{op}$.

We first introduce a very useful object.

Definition 4.2 ($\mathcal{D}_{X\to Y}$): Define the **transfer module** $\mathcal{D}_{X\to Y}$ to be ${}^\circ f^*(\mathcal{D}_Y)$ (i.e., literally the pullback as a quasicoherent $\mathcal{O}$-module):

$$\mathcal{D}_{X\to Y} \coloneqq \mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_Y)} f^{-1}(\mathcal{D}_Y).$$

This has the structure of a $\left(\mathcal{D}_X, f^{-1}(\mathcal{D}_Y)\right)$-bimodule.

Remark 4.3: The functor ${}^\circ f^* : \mathcal{M} \mapsto \mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_Y)} f^{-1}(\mathcal{M})$, of $\mathcal{O}$-modules, also extends to a functor of $\mathcal{D}$-modules.
In other words, let $\mathcal{M}$ be a left $\mathcal{D}_Y$-module, i.e., an $\mathcal{O}_Y$-module with an integral connection. Then a priori, ${}^\circ f^*(\mathcal{M})$ is just an $\mathcal{O}_X$-module, but there's a canonical way to extend the integral connection on $\mathcal{M}$ to an integral connection on ${}^\circ f^*(\mathcal{M})$, so that ${}^\circ f^*(\mathcal{M})$ has a canonical $\mathcal{D}_X$-module structure. This is a right exact functor $\mathcal{D}_Y\text{−mod} \to \mathcal{D}_X\text{−mod}$.

From now on, we'll mainly define the functors at the level of derived categories.

4.1 Pullback functor

In Remark 4.3, we noted that the classical pullback functor ${}^\circ f^* : \mathcal{O}_Y-\text{mod} \to \mathcal{O}_X-\text{mod}$ is right-exact. Therefore we can take the left derived functor:

Definition 4.4 ($\mathbf{L}^\circ f^*$): Define the exact functor $\mathbf{L}^\circ f^* : D^-(\mathcal{D}_Y) \to D^-(\mathcal{D}_X)$ to send

$$u \mapsto \mathcal{D}_{X\to Y} \otimes^{\mathbb{L}}_{f^{-1}(\mathcal{D}_Y)} f^{-1}(u).$$

Note that this functor is also a functor $D^b(\mathcal{D}_Y) \to D^b(\mathcal{D}_X)$ and $D^b_{qc}(\mathcal{D}_Y) \to D^b_{qc}(\mathcal{D}_X)$ as well.

Proposition 4.5: Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be maps of smooth varieties. Then we have canonical isomorphisms

$$\mathcal{D}_{X\to Z} \simeq \mathcal{D}_{X\to Y} \otimes^{\mathbb{L}}_{f^{-1}(\mathcal{D}_Y)} f^{-1}(\mathcal{D}_{X\to Y}) \simeq \mathcal{D}_{X\to Y} \otimes_{f^{-1}(\mathcal{D}_Y)} f^{-1}(\mathcal{D}_{X\to Y}).$$

As an immediate corollary, we have a natural isomorphism of functors

$$\mathbf{L}^\circ(gf)^* \simeq \mathbf{L}^\circ f^* \circ \mathbf{L}^\circ g^*.$$

Proof sketch. First use the fact that in the definition of $\mathcal{D}_{X\to Y}$, we can replace the tensor product with a derived tensor product, since $\mathcal{D}_Y$ is a flat $\mathcal{O}_Y$-module. Then the first isomorphism is a direct computation in the derived categories. The second isomorphism is just because $\mathcal{D}_{X\to Z}$ is a complex concentrated in degree 0, hence the derived tensor product is just an ordinary tensor product. □

Actually, when we pull back in derived categories, due to differences in dimension, we don't actually just want to use $\mathbf{L}^\circ f^*$. We want some shifted version of it:

Definition 4.6 ($f^\dagger$): Define

$$f^\dagger := \mathbf{L}^\circ f^*[\dim X - \dim Y] : D^b(\mathcal{D}_Y) \to D^b(\mathcal{D}_X).$$

4.2 Pushforward functor

The pushforward functor is more naturally defined for right $\mathcal{D}$-modules, because $\mathcal{M} \mapsto \mathcal{M} \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}$ is a right $f^{-1}(\mathcal{D}_Y)$-module, hence ${}^\circ f_*(\mathcal{M} \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y})$ is a right $\mathcal{D}_Y$-module. However, here we have a left exact functor (${}^\circ f_*$) being composed with a right exact functor (the tensor product). This won't give us any good properties - we need exact functors to have something defined on the level of derived categories - so we just derive everything.

Definition 4.7 (f_+): Define the exact functor $f_+ : D^b(\mathcal{D}_X^{op}) \to D^b(\mathcal{D}_Y^{op})$ to send

$$u \mapsto \mathbf{R}^\circ f_* \left(u \otimes^{\mathbb{L}}_{\mathcal{D}_X} \mathcal{D}_{X\to Y}\right).$$

The corresponding functor for left $\mathcal{D}$-modules is obtained by translating between left and right $\mathcal{D}$-modules.

Once again, we want to know that the pushforward functor behaves well with composition.

Proposition 4.8: Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be maps of smooth varieties. Then we have a natural isomorphism of functors

$$(g \circ f)_+ \simeq g_+ \circ f_+.$$

Proof. It's once again a computation, and using the projection formula. ◻

Note that f_+ preserves quasicoherence, i.e., f_+ also induces a functor $D^b_{qc}(\mathcal{D}_X) \to D^b_{qc}(\mathcal{D}_Y)$.

4.3 Explicit computations

Here, I'll explicitly compute certain cases of the previous functors/objects for future use. While all of these will be used in these notes, some of them will just be referred to briefly without elaborating, so your interest level in actually understanding the proofs in this subsection may vary.

Lemma 4.9: For an open embedding $f : U \hookrightarrow X$, then $f^\dagger$ is just the restriction to the open subset U, $(-)|_U$.

Proof. Recall that $f^\dagger := \mathbf{L}^\circ f^*[\dim X - \dim U]$. But $\dim X = \dim U$, and ${}^\circ f^*$ (the quasicoherent pullback) is exact and equal to the restriction to U, so $f^\dagger = {}^\circ f^* = (-)|_U$. ◻

Example 4.10 ($\mathcal{D}_{X\to Y}$ for closed embeddings): Let $i : X \hookrightarrow Y$ be a closed embedding. Now cover Y by suitable affine opens, such that on each affine open, there are coordinates $x_1, \ldots, x_n, y_{n+1}, \ldots, y_m$ such that X is defined by the vanishing of the y_i's. (This can alwaus be done, for example by looking at the regular local rings and taking the distinguished affine opens corresponding to the elements we need to invert.) Then we have:

- $\mathcal{O}_X = \mathbb{C}[x_1, \ldots, x_n]$
- $\mathcal{D}_X = \mathbb{C}[x_1, \ldots, x_n, \partial_{x_1}, \ldots, \partial_{x_n}]$
- $\mathcal{O}_Y = \mathbb{C}[x_1, \ldots, y_m]$
- $\mathcal{D}_Y = \mathbb{C}[x_1, \ldots, y_m, \partial_{x_1}, \ldots, \partial_{y_m}]$

Now we just compute:

$$\begin{aligned}
\mathcal{D}_{X\to Y} &= \mathcal{O}_X \otimes_{i^{-1}(\mathcal{O}_Y)} i^{-1}(\mathcal{D}_Y), \\
&= \mathbb{C}[x_1, \ldots, x_n] \otimes_{\mathbb{C}[x_1,\ldots,y_m]} \mathbb{C}[x_1, \ldots, y_m, \partial_{x_1}, \ldots, \partial_{y_m}], \\
&= \mathcal{D}_Y/(y_{n+1}, \ldots, y_m)\mathcal{D}_Y, \\
&= \mathcal{D}_X[\partial_{y_{n+1}}, \ldots, \partial_{y_n}].
\end{aligned}$$

This is a $(\mathcal{D}_X, i^{-1}\mathcal{D}_Y)$-bimodule, and in particular is a free $\mathcal{D}_X$-module.

Example 4.11 (f_+ under closed embeddings): Let $i : X \hookrightarrow Y$ be a closed embedding, and $\mathcal{M}$ a left $\mathcal{D}_X$-module. We wish to compute $i_+(\mathcal{M})$.

- Recall that $i_+(\mathcal{M}) = \mathbf{R}^\circ i_* \left(\mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{D}_X} \mathcal{D}_{X\to Y}\right)$, at least if $\mathcal{M}$ were a right $\mathcal{D}$-module and we wanted to obtain a right $\mathcal{D}$-module.
- Since $^\circ i_*$ is exact, $\mathbf{R}^\circ i_* = {}^\circ i_*$.
- Since $\mathcal{D}_{X\to Y}$ is a free $\mathcal{D}_X$-module, the derived tensor product is the same as the ordinary tensor product.

Let τ denote the map switching left and right $\mathcal{D}$-modules from §3.3. Then what we actually have is

$$i_+\mathcal{M} = \tau\left(\tau(\mathcal{M}) \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}\right).$$

As a sheaf, this is just $\mathcal{M} \otimes_{\mathbb{C}} \mathbb{C}[\partial_{y_{n+1}}, \ldots, \partial_{y_m}]$. Now let's examine the $\mathcal{D}_Y$-module structure. Let a general element of $i_+\mathcal{M}$ be $u \otimes \partial_y^\beta$, where $\partial_y^\beta = \prod_{i=n+1}^m \partial_{y_i}^{\beta_i}$.

- First, we have $\mathcal{D}_X \subset \mathcal{D}_Y$ as a subalgebra. Now the first τ implies that $\varphi \in \mathcal{D}_X$ acts by $\tau(\varphi)$ on the right of $\tau(\mathcal{M}) \otimes_{\mathbb{C}} \mathbb{C}[\partial_{y_{n+1}}, \ldots, \partial_{y_m}]$. The $\tau(\varphi)$ clearly commutes with the $\mathbb{C}[\partial_{y_{n+1}}, \ldots, \partial_{y_m}]$, hence acts on $\tau(\mathcal{M})$ by $\tau^2(f) = f \curvearrowright \mathcal{M}$. So $\mathcal{D}_X \subset \mathcal{D}_Y$ acts in the standard way on $i_+\mathcal{M}$: we have

$$\varphi \cdot (u \otimes \partial_y^\beta) = (\varphi \cdot u) \otimes \partial_y^\beta.$$

- Next, we check the action of some y_i. First y_i acts on $i_+\mathcal{M}$ by $\tau(y_i) = y_i$ acting on the right on $\tau(\mathcal{M}) \otimes_{\mathbb{C}} \mathbb{C}[\partial_{y_{n+1}}, \ldots, \partial_{y_m}]$. Now $\partial_y^\beta y_i = \beta_i \partial_y^{\beta - e_i}$ because $\partial_y y = y\partial_y + 1$ and $y\partial_y$ is in the ideal $(y_{n+1}, \ldots, y_m)\mathcal{D}_Y$; if $\beta_i = 0$ then y_i commutes with it and sits on the left, which is in the ideal $(y_{n+1}, \ldots, y_m)\mathcal{D}_Y$, hence is zero, agreeing with the formula anyway. So

$$y_i \cdot (u \otimes \partial_y^\beta) = \beta_i u \otimes \partial_y^{\beta - e_i}.$$

- Lastly, we check the action of some ∂_{y_i}. First ∂_{y_i} acts on $i_+\mathcal{M}$ by $\tau(\partial_{y_i}) = -\partial_{y_i}$ acting on the right on $\tau(\mathcal{M}) \otimes_{\mathbb{C}} \mathbb{C}[\partial_{y_{n+1}}, \ldots, \partial_{y_m}]$. Then it commutes with ∂_y^β, so we just get that

$$\partial_{y_i} \cdot (u \otimes \partial_y^\beta) = -u \otimes \partial_y^{\beta + e_i}.$$

Example 4.12 (f_+ under open embeddings): Let $f : X \to Y$ be an open embedding, or more generally an étale morphism. Recall that $\mathcal{D}_{X\to Y} = {}^\circ f^* \mathcal{D}_Y$. But $\mathcal{D}_Y$ is generated by $\mathcal{O}_Y$ and $\mathcal{T}_X = \Omega_Y^\vee$, subject to certain relations. Under étale maps, $^\circ f^* \mathcal{O}_Y = \mathcal{O}_X$ and $^\circ f^* \Omega_Y = \Omega_X$, and the relations pull back as well, so $^\circ f^* \mathcal{D}_Y$ is just the sheaf generated by $\mathcal{O}_X$ and $\Omega_X^\vee$, subject to the same relations - this is exactly $\mathcal{D}_X$. So $\mathcal{D}_X \xrightarrow{\sim} \mathcal{D}_{f:X\to Y}$. Furthermore, $\omega_X = {}^\circ f^* \omega_Y$, so the tensor with ω cancels out in the left-right equivalence, so what we obtain is that

$$f_+ = \mathbf{R}^\circ f_*.$$

4.4 Some general results

These are mostly technical results for later use; feel free to skip if skimming for main ideas.

Theorem 4.13 (proper pushforward preserves coherence): Let $f : X \to Y$ be a proper morphism of smooth irreducible varieties. Then f_+ sends $D^b_{coh}(\mathcal{D}_X) \to D^b_{coh}(\mathcal{D}_Y)$.

Proof.

- Our goal is to show that for $u \in D^b_{coh}(\mathcal{D}_X)$, then $\mathcal{H}^q(f_+(u))$ is a coherent $\mathcal{D}_Y$-module for each q.
- We can use truncation functors to reduce to the case of u being a coherent $\mathcal{D}_X$-module, i.e. a complex concentrated in a single degree.
- Left-right equivalence (3.26) preserves coherence, so we switch to *right* $\mathcal{D}$-modules (as they're cleaner to work with when using the f_+ functor).

- First, we show that every *induced coherent* right $\mathcal{D}_X$-module $\mathcal{M}$ satisfies $f_+(\mathcal{M}) \in D^b_{coh}(\mathcal{D}_Y^{op})$. Induced coherent means that $\mathcal{M} = \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{D}_X$, for some coherent $\mathcal{O}_X$-module $\mathcal{E}$.
  - We just compute and apply projection formula at the end:

$$
\begin{aligned}
f_+(\mathcal{M}) &= \mathbf{R}^\circ f_*\left((\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{D}_X) \otimes^{\mathbb{L}}_{\mathcal{D}_X} \mathcal{D}_{X\to Y}\right),\\
&= \mathbf{R}^\circ f_*\left(\mathcal{E} \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{D}_X \otimes^{\mathbb{L}}_{\mathcal{D}_X} \mathcal{O}_X \otimes^{\mathbb{L}}_{f^{-1}(\mathcal{O}_Y)} f^{-1}(\mathcal{D}_Y)\right),\\
&= \mathbf{R}^\circ f_*\left(\mathcal{E} \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{O}_X \otimes^{\mathbb{L}}_{f^{-1}(\mathcal{O}_Y)} f^{-1}(\mathcal{D}_Y)\right),\\
&= \mathbf{R}^\circ f_*\left(\mathcal{E} \otimes^{\mathbb{L}}_{f^{-1}(\mathcal{O}_Y)} f^{-1}(\mathcal{D}_Y)\right),\\
&= \mathbf{R}^\circ f_*\left(\mathcal{E} \otimes^{\mathbb{L}}_{f^{-1}(\mathcal{O}_Y)} f^{-1}(\mathcal{D}_Y)\right),\\
&= \mathbf{R}^\circ f_*(\mathcal{E}) \otimes^{\mathbb{L}}_{\mathcal{O}_Y} \mathcal{D}_Y.
\end{aligned}
$$

  - But $\mathcal{D}_Y$ is a flat $\mathcal{O}_Y$-module, and f is proper, hence $\mathbf{R}^\circ f_*(\mathcal{E}) \in D^b_{coh}(\mathcal{O}_Y)$.
  - It follows that $\mathcal{H}^q(f_+(\mathcal{M})) = \mathcal{H}^q(\mathbf{R}^\circ f_*(\mathcal{E})) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y$, an induced-coherent $\mathcal{D}_Y^{op}$-module.
  - But induced coherent $\mathcal{D}_Y^{op}$-modules are indeed coherent $\mathcal{D}_Y^{op}$-modules, hence f_+ of induced coherent $\mathcal{D}$-modules are indeed coherent.
- Now let $\mathcal{M}$ be arbitrary. Then $\mathcal{M}$ admits a surjection from an induced coherent $\mathcal{D}_X$-module, say $\mathcal{F}$: then we have a short exact sequence

$$0 \to \mathcal{N} \to \mathcal{F} \to \mathcal{M} \to 0,$$

  where $\mathcal{F}$ is induced coherent.
- Then we apply the long exact sequence in cohomology:

$$\to \mathcal{H}^q(f_+(\mathcal{F})) \to H^q(f_+(\mathcal{M})) \to H^{q+1}(f_+(\mathcal{N})).$$

- Now use descending induction on q, the fact that $\mathcal{H}^q = 0$ for $q \gg 0$, and the fact that $\mathcal{H}^q(f_+(\mathcal{F}))$ is coherent (because $\mathcal{F}$ is induced coherent), to conclude that $\mathcal{H}^q(f_+(\mathcal{M}))$ is coherent for all q.

□

Theorem 4.14 (base change): Suppose we have a Cartesian diagram of smooth irreducible varieties

$$
\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"] & Y.
\end{tikzcd}
$$

Then we have an isomorphism of functors

$$g^\dagger \circ f_+ \simeq f'_+ \circ g'^\dagger.$$

This will be rewritten as

$$g^! \circ f_+ \simeq g'_+ \circ g'^!.$$

Proof idea. The idea is to factor g as a composition of a closed embedding and a projection, then use the explicit descriptions of $g^\dagger$ in each of these two cases. For details, see [HTT07, Theorem 1.7.3]. □

Theorem 4.15 (exact triangle): Let $Z \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} U$ be the inclusions of a smooth irreducible subvariety Z and the open complement U into X. Then we have an exact triangle of functors

$$i_+ \circ i^\dagger \to \mathrm{Id} \to j_+ \circ j^\dagger \to .$$

Note that $j^\dagger$ coincides with the restriction to U, i.e., $j^\dagger = (-)|_U$. In the language of the six functors, this will become

$$i_* i^! \to \mathrm{Id} \to j_* j^* \to .$$

Additionally, note that the functor $\mathbf{R}\Gamma_Z \simeq i_+ \circ i^\dagger = i_* i^! = i_! i^!$, where Γ_Z is the local sections functor.

4.5 Duality functor

We next introduce an important involution on $D^b_{coh}(\mathcal{D}_X)$.

Definition 4.16 ($\mathbb{D}_X$): Let X be a smooth irreducible variety of dimension n. Then we define

$$\mathbb{D}_X : u \mapsto \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(u, \mathcal{D}_X) \otimes_{\mathcal{O}_X} \omega_X^{-1}[n],$$

an exact functor $D^b_{coh}(\mathcal{D}_X) \to D^b_{coh}(\mathcal{D}_X)$.

Remark 4.17: Note that we're really just applying $\mathbf{R}\mathcal{H}\mathrm{om}(-, \mathcal{D}_X)$ and shifting it by the dimension of X. But this lands in $D^b_{coh}(\mathcal{D}_X^{op})$, so we apply the left-right equivalence to land back in $D^b_{coh}(\mathcal{D}_X)$.

Remark 4.18: Note that $\mathbb{D}_X$ is an **involution**, i.e., $\mathbb{D}_X^2$ is naturally isomorphic to the identity functor.

Proposition 4.19: If f is a proper morphism, then $(f_+, f^\dagger)$ is an adjoint pair. More specifically, we have a canonical isomorphism

$$\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_Y}(f_+(u), v) \simeq \mathbf{R}f_*\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(u, f^\dagger(v)).$$

Proof. The proof is essentially a computation. Note that in the third step, we use the fact that if f is proper, there's functorial isomorphisms $\mathbb{D}_Y \circ f_+ \simeq f_+ \circ \mathbb{D}_X$, which we refer to [HTT07, Theorem 2.7.2] (also referred to in Proposition 5.12). In the fourth and fifth steps, we apply projection formula. We write:

$$\begin{aligned}
\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_Y}(f_+(u), v) &\simeq \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_Y}(f_+(u), \mathcal{D}_Y) \otimes^{\mathbb{L}}_{\mathcal{D}_Y} v, \\
&\simeq \mathbb{D}_Y(f_+(u)) \otimes^{\mathbb{L}}_{\mathcal{O}_Y} \omega_Y[-\dim Y] \otimes^{\mathbb{L}}_{\mathcal{D}_Y} v, \\
&\simeq f_+(\mathbb{D}_X(u)) \otimes^{\mathbb{L}}_{\mathcal{O}_Y} \omega_Y[-\dim Y] \otimes^{\mathbb{L}}_{\mathcal{D}_Y} v, \\
&\simeq \mathbf{R}f_* \left(\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(u, \mathcal{D}_X) \otimes^{\mathbb{L}}_{\mathcal{D}_X} \mathcal{D}_{X\to Y}\right) \otimes^{\mathbb{L}}_{\mathcal{D}_Y} v[\dim X - \dim Y], \\
&\simeq \mathbf{R}f_* \left(\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(u, \mathcal{D}_X) \otimes^{\mathbb{L}}_{\mathcal{D}_X} \mathcal{D}_{X\to Y} \otimes^{\mathbb{L}}_{f^{-1}(\mathcal{D}_Y)} f^{-1}(v)\right) [\dim X - \dim Y], \\
&\simeq \mathbf{R}f_* \left(\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(u, \mathcal{D}_X) \otimes^{\mathbb{L}}_{\mathcal{D}_X} f^\dagger(v)\right), \\
&\simeq \mathbf{R}f_* \left(\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(u, f^\dagger(v)\right).
\end{aligned}$$

□

Remark 4.20: Quite a lot of arguments will be implied "by duality." By this, we mean that one should conjugate the equation by $\mathbb{D}$. For example, the statements in Proposition 5.13vi are $i^* j_! = 0$ and $i^! j_* = 0$. Once we know one of them - namely, $i^! j_* = 0$ - we just conjugate the equation by $\mathbb{D}$ to obtain

$$0 = \mathbb{D} i^! j_* \mathbb{D} = (\mathbb{D} i^! \mathbb{D})(\mathbb{D} j_* \mathbb{D}) = i^* j_!.$$

These are due to the fact that star and shriek functors are dual to each other, by definition - they are defined as the $\mathbb{D}$-conjugates of each other, see §5.3.

4.6 Kashiwara's equivalence theorem

In ordinary algebraic topology, one can obtain sheaves (i.e., $\mathcal{O}_X$-modules) supported on some closed subset $Z \subset X$ by pushing forward the relevant sheaf from Z. However, one cannot obtain *all* such sheaves: a quasicoherent $\mathcal{O}_X$-module $\mathcal{F}$ is the pushforward of a quasicoherent $\mathcal{O}_Z$-module iff it is killed by the ideal sheaf, $\mathcal{I}_Z \cdot \mathcal{F} = 0$. However, being supported on Z is a weaker condition: this amounts to having a sufficiently high power of the ideal sheaf annihilating $\mathcal{F}$. So these two categories don't coincide.

However, in the world of $\mathcal{D}$-modules, they do.

Theorem 4.21 (Kashiwara's equivalence): Let X be a smooth $\mathbb{C}$-variety and $Z \subset X$ a smooth irreducible closed subvariety, with inclusion map i. The functor $i_+ : \mathcal{D}_Z\text{−mod}^{qc} \to \mathcal{D}_X\text{−mod}^{qc}$ gives an equivalence of categories between $\mathcal{D}_Z\text{−mod}^{qc}$ and the full subcategory of $\mathcal{D}_X\text{−mod}^{qc}$ of sheaves supported on Z:

$$\begin{tikzcd}
\mathcal{D}_Z\text{−mod}^{qc} \arrow[r, "i_+"] \arrow[dr, "\sim"'] & \mathcal{D}_X\text{−mod}^{qc} \\
 & \mathcal{D}_X\text{−mod}_Z^{qc}. \arrow[u, hook]
\end{tikzcd}$$

The inverse functor is given by $i^\dagger$.

Proof sketch. First we deal with the case of X being affine, and coordinates so that Z is the vanishing of a single coordinate. Then we use the explicit description of $i_+\mathcal{M}$ to see that $\operatorname{supp}(i_+\mathcal{M}) \subseteq Z$. Next, we can explicitly construct an inverse functor to i_+, which after careful consideration, turns out to be exactly $i^\dagger$. The general case follows more or less formally, by covering by suitable affine opens. For more details, see [Mus, Proof of Theorem 6.20, Proposition 6.24]. □

Remark 4.22: It's not hard to see that i_+ also induces equivalences between the quasicoherent and coherent subcategories on both sides (as well as in the derived categories).

In this specific instance of pushforward by a closed embedding, we might wonder how the characteristic variety changes.

Proposition 4.23: Let $i : Z \hookrightarrow X$ be a closed embedding of smooth irreducible varieties of codimension r. We have a surjective morphism $\varphi : T^*X|_Z \to T^*Z$ of vector bundles over Z, with fiber $\mathbb{A}^r$. We also have a closed embedding $\psi : T^*X|_Z \hookrightarrow T^*X$.
For every coherent $\mathcal{D}_Z$-module $\mathcal{M}$, then

$$\operatorname{Char}(i_+\mathcal{M}) = \psi\left(\varphi^{-1}(\operatorname{Char}(\mathcal{M}))\right).$$

In particular, $\dim(i_+\mathcal{M}) = \dim(\mathcal{M}) + r$.

$$\begin{tikzcd}
 & T^*X|_Z \arrow[r, hook, "\psi"] \arrow[d, two heads, "\varphi"'] & T^*X & \operatorname{Char}(i_+\mathcal{M}) \arrow[l, hook'] \\
\operatorname{Char}(\mathcal{M}) \arrow[r, hook] \arrow[urrr, dashed, bend right, mapsto] & T^*Z & &
\end{tikzcd}$$

Proof sketch. The first statement can be checked locally and using the explicit description of $i_+\mathcal{M}$, then constructing a good filtration on $i_+\mathcal{M}$ by $\psi_*\varphi^*$ applied to a good filtration on $\mathcal{M}$. The second statement follows from the first. □

4.7 Proof of Bernstein's inequality

Recall Bernstein's inequality (3.25): the dimension of any nonzero coherent $\mathcal{D}_X$-module is at least $\dim X$ (and even every irreducible component of $\operatorname{Char}(\mathcal{M})$ also satisfies this).

Proof of Theorem 3.25, Bernstein's inequality.

- We'll induct on $n = \dim X$. The case $n = 0$ is trivial.
- The claim is local, so we may cover X by finitely many affine opens, and reduce to the case where $X = \operatorname{Spec} A$ is affine. Let $R = D_{\operatorname{Spec} A}$ and M the associated module to $\mathcal{M}$.
- We have a finite decreasing filtration of M (the Gabber filtration) by codimension, $M = C^0(M) \supseteq C^1(M) \supseteq \cdots \supseteq C^{2n+1}(M) = 0$; the successive quotients are pure R-modules of codimension i.
- Since $\operatorname{Char}(M)$ is just the union of the characteristic varieties of the quotients C^i/C^{i+1}, we may assume M is a pure R-module. Let us now assume that $\dim M < n$, for the sake of contradiction.
- Let $\pi : T^*X \to X$ be the canonical projection. Then $Z = \pi(\operatorname{Char}(M))$ is a proper closed subset of X of codimension r. We may shrink X to an open subset $U \subset X$ so that $Z \cap U$ is (nonempty) smooth and irreducible; so that we may assume Z is smooth and irreducible.
- Now Kashiwara's equivalence theorem 4.21 implies that $\mathcal{M} \simeq i_+\mathcal{N}$ for some nonzero $\mathcal{D}_Z$-module $\mathcal{N}$.
- Since $\dim Z < \dim X$, we can apply the inductive hypothesis (in the first step) to see that $\dim \mathcal{N} \geq n - r$.
- It remains to use Proposition 4.23 to see how the dimension of a $\mathcal{D}$-module changes under the pushforward functor; we have $\dim \mathcal{M} = \dim i_+\mathcal{N} = \dim \mathcal{N} + r \geq n$, contradiction.

□

5 Holonomic $\mathcal{D}$-modules and the six functors

As in the case with quasicoherent $\mathcal{O}_X$-modules, we can define certain functors on general $\mathcal{D}$-modules (such as pushforward and pullback), but we can't always define some other functors, such as upper shriek and lower shriek (only in certain cases). What is the setting where we can? It turns out that a class of $\mathcal{D}$-modules called *holonomic $\mathcal{D}$-modules* is a setting where we can define more functors that fit into a nice formalism called the six-functor formalism. Essentially, we have "enough" functors which interact in very nice ways, allowing us to approach many problems formally.

5.1 Holonomic $\mathcal{D}$-modules

Recall that Bernstein's inequality (3.25) says that $\dim \mathcal{M} \geq \dim X$ for any nonzero coherent $\mathcal{D}_X$-module $\mathcal{M}$. Holonomic $\mathcal{D}$-modules are those of minimal dimension.

Definition 5.1: Let X be a smooth irreducible variety of dimension n. A **holonomic** $\mathcal{D}_X$-module is a $\mathcal{D}$-module of dimension equal to n.

Example 5.2: If a $\mathcal{D}_X$-module is coherent as an $\mathcal{O}_X$-module, then its characteristic variety is the 0-section of T^*X, hence has dimension n, hence is holonomic.
In fact, the converse is also true: a coherent $\mathcal{D}_X$-module whose characteristic variety is the 0-section is necessarily a coherent $\mathcal{O}_X$-module.

It's easy to check that the category of holonomic $\mathcal{D}_X$-modules, which we'll denote by $\mathsf{Hol}(\mathcal{D}_X)$, is an abelian category which is closed under extensions, and furthermore this is a finite-length category. Therefore, the long exact sequence in cohomology (associated to an exact triangle) implies that we have a triangulated subcategory $D^b_{hol}(\mathcal{D}_X) \subset D^b_{coh}(\mathcal{D}_X)$, whose objects are those complexes whose cohomology objects are all holonomic $\mathcal{D}_X$-modules. It's also not hard to see that holonomicity is preserved under the left-right equivalence, but actually something more specific is true. It turns out that

$$\mathcal{M} \text{ holonomic} \iff \dim \mathcal{M} \leq n \iff \operatorname{codim} \mathcal{M} \geq n \iff \mathcal{E}xt^i(\mathcal{M}, \mathcal{D}_X) \neq 0 \text{ only for } i = n.$$

We'd like a characterization of holonomic objects. The next theorem will give us such a characterization. (Actually, the proof of this requires Proposition 5.8, so it should really go after that, but I'm listing it here because the result seems to fit in this subsection better. Logically, however, it doesn't need anything after Proposition 5.8 and nothing before Proposition 5.8 needs this result in their proof.)

Theorem 5.3: Let X be a smooth irreducible variety, and let $w \in D^b_{coh}(\mathcal{D}_X)$. The following are equivalence:

(i) $w \in D^b_{hol}(\mathcal{D}_X)$.

(ii) For every $x \in X$, letting $i_x : \{x\} \hookrightarrow X$ be the inclusion, then $\dim_{\mathbb{C}} \mathcal{H}^q\left(i_x^\dagger(w)\right) < \infty$ for all $q \in \mathbb{Z}$.

(iii) There is a sequence of closed subsets

$$X = Z_0 \supseteq Z_1 \supseteq \cdots \supseteq Z_r \supseteq Z_{r+1} = \emptyset$$

such that for all $0 \leq j \leq r$, the locally closed subset $W_j := Z_j \setminus Z_{j+1}$ is smooth, and letting $i_j : W_j \hookrightarrow X$ be the inclusion map, then all $\mathcal{H}^q\left(i_j^\dagger(w)\right)$ are $\mathcal{O}_{W_j}$-coherent.

Proof.

- i $\Longrightarrow$ ii is clear: on a point, holonomic=coherent-finite-dimensional.
- iii $\Longrightarrow$ ii is also clear; pullback preserves holonomicity by Proposition 5.8.
- Now we do ii $\Longrightarrow$ iii,i. Our strategy will be to construct Z_i satisfying iii but that *also* satisfies $i^\dagger_{X\setminus Z_i}(w)$ is holonomic (this will obviously imply i, since that's just the statement for $Z_{r+1} = \emptyset$).
- To do this, we want to show that if $Z \subset X$ is a closed subset such that $i^\dagger_{X\setminus Z}(w)$ is holonomic, then there exists a closed $Z' \subsetneq Z$ such that $Z \setminus Z'$ is smooth, and $i^!_{Z\setminus Z'}(w)$ is $\mathcal{O}_{Z\setminus Z'}$-coherent. Then Noetherian induction shows that this process terminates and gives us the sequence of closed subsets that we need.
  - Pick an open $U \subset X$, such that $U \cap Z$ is a smooth irreducible open subset of Z. Let $i : U \cap Z \hookrightarrow U$ and $j : U \setminus Z \hookrightarrow U$ be the closed and open embeddings.
  - We have the exact triangle (4.15)

    $$i_+ i^\dagger(w|_U) \longrightarrow w|_U \longrightarrow j_+(w|_{U\setminus Z}).$$

  - Since w is coherent, so is $w|_U$. Also, by hypothesis, $w|_{X\setminus Z}$ is holonomic, and j_+ preserves holonomicity (5.7), hence coherent.
  - Since the second and third terms are coherent, so is the first one. But $i_+ i^\dagger(w|_U)$ coherent implies that $i^\dagger(w|_U)$ is coherent on $U \cap Z$.
  - There is a nonempty affine open subset $f : W \hookrightarrow U \cap Z$ such that $f^\dagger i^\dagger(w|_U)$ is a free $\mathcal{O}_W$-module.
    - More cleanly, if $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module (for X smooth irreducible), then there's a nonempty open affine $U \subset X$ such that $\mathcal{M}|_U$ is a free $\mathcal{O}_U$-module, possibly of infinite rank.
    - Replace X with an affine open $\operatorname{Spec} R$, and let S be the algebra $\mathcal{D}_X$.
    - Then $\operatorname{Gr}^F_\bullet(\mathcal{M})$ is finitely generated over $\operatorname{Gr}^F_\bullet(S)$.
    - Generic freeness implies that $\operatorname{Gr}^F_\bullet(\mathcal{M})$ is generically free over R, hence there exists $f \in R$ such that $\operatorname{Gr}^F_\bullet(\mathcal{M})_f$ is free over R_f.
    - Now lift this basis to that of $\mathcal{M}_f$ over R_f, which makes it free.
  - So we can replace U with a smaller open, so that $U \cap Z$ is affine and $i^!(w|_U)$ is a free $\mathcal{O}_{U\cap Z}$-module. Let $d = \dim U \cap Z$.
  - Then $\mathcal{H}^q(i_x^\dagger(w)) \simeq \mathcal{H}^{q+d}(i^\dagger(w|_U)) \otimes_{\mathbb{C}} \mathbb{C}(x)$. Since the LHS is finite-dimensional by assumption, then $i^\dagger w|_U$ is a coherent $\mathcal{O}_{Z\cap U}$-complex, and hence $i^\dagger w|_U$ is holonomic (see Example 5.2).
  - Now take $Z' := Z \setminus U$, $W := Z \cap U$. We just showed that $i_W^\dagger(w)$ is holonomic. It remains to show that $i^\dagger_{X\setminus Z'}(w)$ is also holonomic. But set-theoretically, $X \setminus Z' = (X \setminus Z) \sqcup (Z \cap U)$, as a union of an open and closed subset of $X \setminus Z'$. The (shriek-)restriction of w to each of these is holonomic; the latter we just proved, and the former by assumption. Thus the exact triangle tells us that the (shriek-)restriction to the entire thing, $i^\dagger_{X\setminus Z'}(w)$ is also holonomic.

□

Corollary 5.4: If $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module, then there is a nonempty open subset $U \subset X$ for which $\mathcal{M}|_U$ is $\mathcal{O}_U$-coherent.

Proof. Just take $U := W_1 = Z_0 \setminus Z_1 = X \setminus Z_1$. Since this is open, we have $i^\dagger = (-)|_U$ (in the language of the six functors, $i^! = i^* = (-)|_U$), and U is clearly nonempty. The claim then follows from part iii of the theorem. □

Remark 5.5: The condition that w is *holonomic* in Theorem 5.3 is crucial. For example, take $w = \mathcal{D}_X$, which is coherent but not holonomic. Then take $X = \mathbb{A}^n$, and $x \in X$ some arbitrary point, say the origin, with i_x the inclusion map. Then

$$\begin{aligned} i_x^\dagger(w)[n] &= \mathbf{L}^\circ i_x^*(\mathcal{D}_X), \\ &= \mathcal{D}_{x\to X} \otimes^l l_{i_x^{-1}(\mathcal{D}_X)} i_x^{-1}(\mathcal{D}_X), \\ &= \mathcal{D}_{x\to X}, \\ &= \mathcal{O}_x \otimes_{i_x^{-1}(\mathcal{O}_X)} i_x^{-1}(\mathcal{D}_X), \\ &= \mathbb{C}[\partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_n}], \end{aligned}$$

viewed as a sheaf (i.e., a chain complex concentrated in a single degree) over a point. But this is infinite-dimensional (in at least one of the degrees), thus breaking part ii of the theorem.

Remark 5.6: So a holonomic object is not actually truly $\mathcal{O}$-coherent; but it is $\mathcal{O}$-coherent on a certain stratification. The third condition is how we can build up a holonomic object via exact triangles with open/closed subsets, while the second condition is useful if we want to check that something is holonomic.

5.2 Preservation of holonomicity

We want to show that the two functors we constructed, f_+ and $f^\dagger$, preserve holonomicity. Naturally, we'll use the fact that any f can be factored into an open embedding and a closed embedding (or some variant), so we'll handle these two cases separately and in stages. If you are impatient, feel free to skip to the end of the section, Theorem 5.10.

Lemma 5.7: Let X be a smooth irreducible variety, U an open subset of X, with inclusion map $j : U \hookrightarrow X$. Then j_+ maps $D^b_{hol}(\mathcal{D}_U) \to D^b hol(\mathcal{D}_X)$.

Proof. I'm going to skip this one, because Mustata proves this using b-functions, which I don't feel like introducing here just for the sake of proving preservation of holonomicity. Feel free to consult [HTT07, Theorem 6.50]. □

Proposition 5.8: Let $f : X \to Y$ be a morphism of smooth irreducible varieties. Then $f^\dagger$ sends $D^b_{hol}(\mathcal{D}_Y) \to D^b_{hol}(\mathcal{D}_X)$.

Proof.

- Since $f^\dagger$ is just a shift of $\mathbf{L}^\circ f^*$, it's equivalent to prove it for the latter.
- Factor f as $X \overset{i}{\hookrightarrow} X \times Y \overset{p}{\hookrightarrow} Y$, where i is a closed embedding and p is a projection, i.e. a proper map. Since $\mathbf{L}^\circ f^* = \mathbf{L}^\circ i^* \circ \mathbf{L}^\circ p^*$, it suffices to prove it in each case.
- First suppose $f = i$ is a closed embedding.
  - The exact triangle tells us that for any $v \in D^b_{hol}(\mathcal{D}_Y)$, we have

    $$i_+ i^! v \to v \to j_+(v|_U) \to$$

    is exact.
  - Clearly v is holonomic; also, $v|_U$ is holonomic, and Lemma 5.7 tells us that j_+ of a holonomic thing is holonomic. Thus the latter two terms of the exact triangle are holonomic, hence $i_+ i^! v$ is also holonomic.
  - Now Proposition 4.23 tells us that $\dim(i_+\mathcal{M}) = \dim(\mathcal{M}) + \text{codim}$. Thus for each cohomology object $\mathcal{M}$ of $i_+ i^! v$, we have $\dim(i_+ i^! \mathcal{M}) = \dim(i^! \mathcal{M}) + \dim Y - \dim X$, and the left hand side is equal to $\dim Y$ because it's holonomic. This implies that $\dim i^! \mathcal{M} = \dim X$, hence $i^! v$ is holonomic.

- Now suppose $f = p : X \times Y \to Y$ is a projection. We may assume (for example, by using truncation functors) that we are just dealing with an honest $\mathcal{D}$-module $\mathcal{M}$ (concentrated in degree 0), rather than a complex.
    - We may also assume that X, Y are affine. $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} S$.
    - Then $p^*\mathcal{M} \simeq (R \otimes_{\mathbb{C}} S) \otimes_S \mathcal{M} \simeq R \otimes_{\mathbb{C}} \mathcal{M}$.
    - Take a good filtration $F_\bullet\mathcal{M}$ of $\mathcal{M}$. Then we can construct a good filtration of $p^*\mathcal{M} \simeq R \otimes_{\mathbb{C}} \mathcal{M}$ by taking the filtration to be $R \otimes_{\mathbb{C}} F_\bullet\mathcal{M}$. It follows then that $\operatorname{Char}(p^*\mathcal{M}) = X \times \operatorname{Char}(\mathcal{M})$.
    - It follows that the dimension of the characteristic variety of $p^*\mathcal{M}$ is $\dim X + \dim \operatorname{Char}(\mathcal{M}) = \dim X + \dim Y$, which indeed matches $\dim X \times Y$. Thus $p^*\mathcal{M}$ is also holonomic.

□

At this point, you can go back and read the proof to Theorem 5.3 if you haven't already, since it uses Proposition 5.8. Or I guess you can just keep going.

Finally, it remains to prove that pushforward preserves holonomicity.

Proposition 5.9: Let $f : X \to Y$ be a morphism of smooth irreducible varieties. Then f_+ maps $D^b_{hol}(\mathcal{D}_X) \to D^b_{hol}(\mathcal{D}_Y)$.

Proof.

- First we want to reduce to the case of X, Y being complete varieties.
    - Theorems of Nagata and Hironaka allow us to find open embeddings $j_X : X \hookrightarrow \overline{X}$, $j_Y : Y \hookrightarrow \overline{Y}$, along with a map $\overline{f} : \overline{X} \to \overline{Y}$, where $\overline{X}, \overline{Y}$ are complete, smooth, and irreducible.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "j_X"'] & Y \arrow[d, hook, "j_Y"] \\ \overline{X} \arrow[r, "\overline{f}"] & \overline{X} \end{tikzcd}$$

    - We can find these maps such that $\overline{f} \circ j_X = j_Y \circ f$. In particular,

$$\overline{f}_+ \circ j_{X+} = j_{Y+} \circ f_+.$$

    - Note that $f_+ = (j_{Y+} \circ f_+(-))\,|_Y$, so it suffices to show that $\overline{f}_+ \circ j_{X+}$ sends holonomic to holonomic (as we can then restrict to Y, which preserves holonomicity by Proposition 5.8, and recover f_+).
    - Since $j_X : X \hookrightarrow \overline{X}$ is an open embedding, then j_{X+} already sends holonomic to holonomic, as in Lemma 5.7. So it suffices to show that $\overline{f}_+$ sends holonomic to holonomic; thus we've reduced to the case of f being a map of complete smooth irreducible varieties.
- Assume $f : X \to Y$ is a map of complete varieties.
- Factor $f = p \circ i$, $X \overset{i}{\hookrightarrow} X \times Y \overset{p}{\hookrightarrow} Y$, a composition of a closed embedding with a proper map.
- Then $f_+ = p_+ \circ i_+$.
- We already know that i_+ maps holonomic to holonomic; for example, by calculating the dimension of the characteristic variety of $i_+\mathcal{M}$, as in Proposition 4.23.
- So it suffices to assume $f = p$ is proper.
    - Then Theorem 4.13 implies that proper pushforward preserves coherence, so if $v \in D^b_{hol}(\mathcal{D}_{X \times Y})$, then $p + (v) \in D^b_{coh}(\mathcal{D}_Y)$.
    - Now Theorem 5.3tells us that to verify holonomicity of $p_+(v)$, we just need to check that all fibers are finite-dimensional.
    - So we apply base change (4.14) to the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X \times Y \arrow[d, "p"] \\ \{y\} \arrow[r] & Y. \end{tikzcd}$$

This will allow us to compute the fiber at $\{y\} \hookrightarrow Y$ of $p_+(v)$ by pulling back v along $X \to X \times Y$, then pushing forward along $X \to \{y\}$.

- The pullback along $X \to X \times Y$ preserves holonomicity (5.8), and the pushforward along the proper map $X \to \{y\}$ sends coherent to coherent. But $\{y\}$ is a point, so coherent=holonomic=finite-dimensional, hence the fibers are finite-dimensional, hence $p_+(v)$ is holonomic.

□

To summarize:

Theorem 5.10: Let $f : X \to Y$ be a morphism of smooth irreducible varieties. Then f_+ maps $D^b_{hol}(\mathcal{D}_X) \to D^b_{hol}(\mathcal{D}_Y)$ and $f^\dagger$ maps $D^b_{hol}(\mathcal{D}_Y) \to D^b_{hol}(\mathcal{D}_X)$.

5.3 The six functors

To set up the six-functor formalism, we need… six functors, acting on the bounded derived category of holonomic $\mathcal{D}$-modules.
First, we need two standard functors:

- The derived tensor product, $- \otimes^{\mathbb{L}} -$.
- The (derived) internal hom, $\mathbf{R}\mathcal{H}\mathrm{om}(-,-)$.

We won't actually discuss these too much, though. They play more or less standard roles.

Next, to a map $f : X \to Y$ (of smooth irreducible varieties), we want to associate to it four functors: $f_*, f^*, f_!, f^!$ between the bounded derived categories of holonomic $\mathcal{D}$-modules. We only really defined two functors before: f_+ and $f^\dagger$. To obtain the other two, we conjugate by the duality functor $\mathbb{D}$.

- The lower star functor, $f_* : D^b_{hol}(\mathcal{D}_X) \to D^b_{hol}(\mathcal{D}_Y)$, is just the functor f_+, which we defined earlier in Definition 4.7.
- The lower shriek functor, $f_! : D^b_{hol}(\mathcal{D}_X) \to D^b_{hol}(\mathcal{D}_Y)$, is defined by conjugating by $\mathbb{D}$: we define $f_! := \mathbb{D}_Y \circ f_* \circ \mathbb{D}_X$.
- The upper shriek functor, $f^! : D^b_{hol}(\mathcal{D}_Y) \to D^b_{hol}(\mathcal{D}_X)$, is just the functor $f^\dagger$, defined earlier in Definition 4.6.
- The upper star functor, $f^* : D^b_{hol}(\mathcal{D}_Y) \to D^b_{hol}(\mathcal{D}_X)$, is defined by conjugating by $\mathbb{D}$: we define $f^* := \mathbb{D}_X \circ f^! \circ \mathbb{D}_Y$.

Remark 5.11: Using the earlier facts that $(g \circ f)_* \simeq g_* \circ f_*$ and $(g \circ f)^! \simeq f^! \circ g^!$, along with the fact that $\mathbb{D}^2 \simeq \mathbf{Id}$, we automatically obtain that $(g \circ f)_! \simeq g_! \circ f_!$ and $(g \circ f)^* \simeq f^* \circ g^*$.

5.4 Basic results

Here, we'll quickly summarize basic results regarding the functors we described.

Proposition 5.12: Let $f : X \to Y$ be a morphism of smooth irreducible varieties.
(i) If f is proper, then $f_! \simeq f_*$ canonically.
(ii) The pair $(f_!, f^!)$ is adjoint pair.
(iii) The pair (f^*, f_*) is an adjoint pair.
(iv) If f is an affine morphism, then f_* is right t-exact and $f_!$ is left t-exact.
(v) If f is a finite morphism, then $f_! = f_*$ is t-exact.

Proof. (i) See [HTT07, Theorem 2.7.2].

(ii)
- Use Nagata-Deligne to write $f = p \circ j$ for p a proper morphism and j an open embedding.
- First deal with $f = p$ is a proper morphism.
  - In Proposition 4.19 we showed that $(f_+, f^\dagger) = (f_*, f^!)$ are adjoints.
  - When $f = p$ is proper, $f_! = f_*$, ence $(f^!, f_!)$ are adjoints.

- Next deal with $f = j$ is an open embedding.
  - Since duality is compatible with restriction to open subsets, we have that $f^! = f^* = -|_U$, f_* is the quasicoherent pushforward, and (f^*, f_*) are adjoints in the quasicoherent $\mathcal{O}$-modules case. (See Proposition 5.13ii.)
  - Applying duality shows that $(f^!, f_!)$ are also adjoints when $f = j$.
- So for any such f, then $(f^!, f_!)$ are adjoints.

(iii) Apply duality to the fact that $(f^!, f_!)$ are adjoints. This statement is equivalent (indeed, the dual) to the previous statement.
(iv) f_* is the right adjoint to f^*, which is (a shift of) the left derived functor $\mathbf{L}^\circ f^*$ (which is right t-exact), hence is right t-exact. On the other hand, $f_! = \mathbb{D} f_* \mathbb{D}$, hence is left t-exact.
(v) If f is finite, then it is affine and proper, hence by the previous part $f_* = f_!$ is both right and left t-exact.

□

Now we collect some basic results when f is an open or closed embedding.

Proposition 5.13: Let X be a smooth irreducible variety, $j : U \hookrightarrow X$ be an open embedding, and $i : Z \hookrightarrow X$ the closed embedding of the complement.
(i) j_* is left t-exact and $j_!$ is right t-exact.
(ii) $j^* = j^! = (-)|_U$.
(iii) $j^* j_* \simeq \mathrm{Id}$ and $j^! j_! \simeq \mathrm{Id}$.
Now assume further that Z is smooth.
(iv) $i_! = i_*$ is t-exact.
(v) $i^* i_* \simeq \mathrm{Id}$ and $i^! i_! \simeq \mathrm{Id}$.
(vi) $i^* j_! = 0$ and $i^! j_* = 0$.
(vii) $j^* i_* = 0$.

Proof.
(i) Since $j_* = j_+ = \mathbf{R}^\circ j_*$, we have j_* is a right derived functor and hence left t-exact. The other statement follows by duality.
(ii) We already know that $j^!$ is restriction to U, and duality is compatible with restriction to open subsets.
(iii) The first isomorphism follows from $j_*(-)|_U = \mathrm{Id}$, and the second isomorphism follows from duality.
(iv) We already know that $i_! = i_* = i_+$; furthermore, i_+ is an exact functor on abelian categories, hence is t-exact.
(v) The statement that $i^! i_! \simeq \mathrm{Id}$ is just Kashiwara's equivalence theorem 4.21. The first statement follows by duality.
(vi) To see that $i^! j_* = 0$, we need only note that $i^\dagger j_+ = 0$ as quasicoherent sheaves already. The first statement follows from duality.
(vii) This is saying that for every holonomic $\mathcal{D}_Z$-module $\mathcal{M}$, then $i_+(\mathcal{M})|_U = 0$. But this is already true for every quasicoherent $\mathcal{O}_Z$-module.

□

6 The intermediate extension

Let $f : X \to Y$ be a morphism of smooth irreducible varieties. We wish to define a unique map

$$\beta_f(u) : f_!(u) \to f_*(u)$$

for each $u \in D^b_{hol}(\mathcal{D}_X)$, assembling into a functorial map

$$\beta_f : f_! \to f_*,$$

based on certain desired criteria. First, by Nagata-Deligne, any f can be factored as $f = p \circ j$, where p is a proper map and j is an open embedding, so it suffices to specify such maps on proper maps and open embeddings. For open embeddings, we want $\beta_f(u)$ to be the unique morphism $f_!(u) \to f_*(u)$ whose restriction to $X \subset Y$ is the canonical isomorphism. For proper maps, we want $\beta_f(u)$ to be the functorial isomorphism $f_!(u) \xrightarrow{\sim} f_*(u)$. This now uniquely specifies the map β_f; it remains to check that β_f "composes well," in the sense that

$$\begin{tikzcd}
 & & g_! f_*(u) \arrow[dr, "\beta_g(f_*)"] & & \\
(g \circ f)_!(u) \arrow[r, phantom, "="] & g_! f_!(u) \arrow[ur, "g_!(\beta_f)"] \arrow[rr, "\beta_{g \circ f}"] \arrow[dr, "\beta_g(f_!)"'] & & g_* f_*(u) \arrow[r, phantom, "="] & (g \circ f)_*(u) \\
 & & g_* f_!(u) \arrow[ur, "g_*(\beta_f)"'] & &
\end{tikzcd}$$

commutes. But we'll leave this as details to check for the reader (or you can look at [Mus]).

Remark 6.1: It immediately follows that β_f is compatible with duality, i.e.,

$$\beta_f \circ \mathbb{D}_X = \mathbb{D}_Y \circ \beta_f.$$

6.1 Setup

Now we specialize f to specifically be a locally closed embedding. Let X be a smooth irreducible variety and W a locally closed subscheme which is also smooth and irreducible, and let

$$f : W \hookrightarrow X$$

be the embedding. We will also write f as a composition of a closed and open embedding:

$$W \overset{i}{\hookrightarrow} U \overset{j}{\hookrightarrow} X,$$

where i is a closed embedding and j is an open embedding.

6.2 Definition

Studying the map β_f turns out to be very fruitful. We want to construct one more functor of $\mathcal{D}$-modules, but both f_* and $f_!$ turn $\mathcal{D}$-modules into a chain complex. We rectify this by applying $\mathcal{H}^0$ to get get a map of genuine $\mathcal{D}$-modules:

$$\gamma_f(\mathcal{M}) := \mathcal{H}^0(\beta_f) : \mathcal{H}^0(f_!(\mathcal{M})) \to \mathcal{H}^0(f_*(\mathcal{M})).$$

Once again, this assembles into a functorial morphism

$$\gamma_f : \mathcal{H}^0(f_!) \to \mathcal{H}^0(f_*).$$

Definition 6.2: We define the **intermediate extension** to be the functor $\mathrm{Hol}(\mathcal{D}_W) \to \mathrm{Hol}(\mathcal{D}_X)$

$$f_{!*}(\mathcal{M}) := \mathrm{im}(\gamma_f(\mathcal{M})).$$

Remark 6.3 ($i_{!*}$ and $j_{!*}$): When $f = i$ is a closed embedding, then $\beta_i : i_! \xrightarrow{\sim} i_*$ is an isomorphism, and so $i_{!*} = i_! = i_*$.
This means that to describe $f_{!*}$, we usually just need to describe it when f is an open embedding, as if $f = j \circ i$ then $f_{!*} = j_{!*} \circ i_+$.
On the other hand, when $f = j : U \hookrightarrow X$ is an open embedding, we can construct the map $\beta_j(\mathcal{N}|_U) : j_!(\mathcal{N}|_U) \to j_*(\mathcal{N}|_U)$ for a $\mathcal{D}_X$-module $\mathcal{N}$ via adjointness. The pair of adjoint functors $(j_!, j^!)$ gives us the distinguished morphism $j_! j^! \mathcal{N} \to \mathcal{N}$, adjoint to the identity map $j^! \mathcal{N} \xrightarrow{\text{id}} j^! \mathcal{N}$. Similarly, the pair of adjoint functors (j^*, j_*) gives us the distinguished morphism $\mathcal{N} \to j_* j^* \mathcal{N}$, adjoint to the identity map $j^* \mathcal{N} \to j^* \mathcal{N}$. It turns out that the composition of these maps is precisely β_j:

$$\begin{tikzcd} j_! j^! \mathcal{N} \arrow[r] \arrow[rr, bend left, "\beta_j(\mathcal{N}|_U)"] & \mathcal{N} \arrow[r] & j_* j^* \mathcal{N}. \end{tikzcd}$$

For open embeddings, recall (see Proposition 5.13ii) that $j^! = j^* = (-)|_U$, hence we find that $\beta_j(\mathcal{N}|_U)$ is nothing more than the composition

$$\beta_j(\mathcal{N}|_U) : \quad j_!(\mathcal{N}|_U) \to \mathcal{N} \to j_*(\mathcal{N}|_U),$$

and that $\gamma_j(\mathcal{N}|_U)$ is nothing more than applying $\mathcal{H}^0$ to this composition:

$$\gamma_j(\mathcal{N}|_U) : \quad \mathcal{H}^0(j_!(\mathcal{N}|_U)) \to \mathcal{N} \to \mathcal{H}^0(j_*(\mathcal{N}|_U)).$$

6.3 Main properties

Theorem 6.4: Let $f : W \to X$ be a locally closed embedding of smooth irreducible varieties, and let $\mathcal{M} \in \mathsf{Hol}(\mathcal{D}_W)$.

(i) $f_{!*}$ commutes with duality, i.e., we have isomorphisms $f_{!*} \circ \mathbb{D}_W \simeq \mathbb{D}_X \circ f_{!*}$.

(ii) There are canonical isomorphisms $f^! \circ f_{!*} \simeq \mathrm{Id} \simeq f^* \circ f_{!*}$.

(iii) $f_{!*}$ preserves injections and surjections.

(iv) If $f = j$ (so $W = U$) is an open embedding and $\mathcal{N}$ is a holonomic $\mathcal{D}_X$-module with $\mathcal{N} \subseteq \mathcal{H}^0(j_*(\mathcal{N}|_W))$ (i.e., $\Gamma_{X \setminus U}(\mathcal{N}) = 0$), then $j_{!*}(\mathcal{N}|_U) \subseteq \mathcal{N}$.
In particular, $j_{!*}(\mathcal{M})$ is the smallest $\mathcal{D}_X$-submodule of $\mathcal{H}^0(j_*(\mathcal{M}))$ whose restriction to U is $\mathcal{M}$. Dually, $j_{!*}(\mathcal{M})$ is the smallest $\mathcal{D}_X$-module quotient of $\mathcal{H}^0(j_!(\mathcal{M}))$ whose restriction to U is $\mathcal{M}$.

(v) If $f = j$ (so $W = U$) is an open embedding, then $j_{!*}(\mathcal{M})$ is the unique quasicoherent $\mathcal{D}_X$-module (up to isomorphism) whose restriction to U is $\mathcal{M}$, and has no submodule or quotient module supported on $X \setminus U$.

(vi) If $\mathcal{M}$ is a simple $\mathcal{D}_W$-module, then $f_{!*}(\mathcal{M})$ is a simple $\mathcal{D}_X$-module.
In fact, $f_{!*}(\mathcal{M})$ is the unique simple $\mathcal{D}_X$-submodule of $\mathcal{H}^0(f_*(\mathcal{M}))$, and is the unique simple quotient $\mathcal{D}_X$-module of $\mathcal{H}^0(f_!(\mathcal{M}))$.

(vii) $f_{!*}$ is a fully faithful functor $\mathsf{Hol}(\mathcal{D}_W) \to \mathsf{Hol}(\mathcal{D}_X)$.

Remark 6.5: These properties lend credence to the name "minimal extension." The extending part is clear: the functor $f_{!*}$ is sort of just extending a (holonomic) $\mathcal{D}_W$-module $\mathcal{M}$ to $X \setminus W$; part ii implies that $f_{!*}(\mathcal{M})$ should restrict in both ways to $\mathcal{M}$.
The key is that it should be done *minimally*: parts iv and v imply that while we do need to extend $\mathcal{M}$ to $X \setminus W$ in order to be a true $\mathcal{D}_X$-module, we should do so minimally, without any subs or quotients supported on $X \setminus U$.

Proof of Theorem 6.4.

(i) We need only know that β_f commutes with $\mathbb{D}$ (see Remark 6.1), and that $\mathbb{D}$ is an exact functor, hence γ_f does as well.

(ii)
- Once again, we'll write $f = j \circ i$ as a composition of a closed embedding with an open embedding, and treat both cases separately.
- Suppose $f = j$ is an open embedding.

- Then $j^*\beta_f(u) \simeq u$ by construction.
- Also, $j^! = j^* = (-)|_U$, so both $j^!$ and j^* are left inverses to $j_{!*}$.

- Suppose $f = i$ is a closed embedding.
  - Then $i_{!*} = i_! = i_*$.
  - Now we use Proposition 5.13v to see that i^* and $i^!$ are left inverses of $i_{!*}$.

(iii) This follows from the fact that $\mathcal{H}^0(f_*(-))$ is left exact, and $\mathcal{H}^0(f_!(-))$ is right exact.

(iv)
- Let's assume $\mathcal{N} \subseteq \mathcal{H}^0(j_*(\mathcal{N}|_W))$.
- Let's prove that $j_{!*}(\mathcal{N}|_U) \subset \mathcal{N}$.
  - From Remark 6.3, we have that $\gamma_j(\mathcal{N}|_U)$ factors as
  $$\mathcal{H}^0(j_!(\mathcal{N}|_U)) \to \mathcal{N} \hookrightarrow \mathcal{H}^0(j_*(\mathcal{N}|_U)).$$
  The second map is injective by assumption.
  - Since $j_{!*}(\mathcal{N}|_U)$ is defined to be the image of $\gamma_j(\mathcal{N}|_U)$, it follows that $j_{!*}(\mathcal{N}|_U) \subseteq \operatorname{im}\left(\mathcal{N} \hookrightarrow \mathcal{H}^0(j_*(\mathcal{N}|_U))\right) = \mathcal{N}$.
- Now let's prove $j_{!*}(\mathcal{M})$ is the smallest $\mathcal{D}_X$-submodule of $\mathcal{H}^0(j_*(\mathcal{M}))$ whose restriction to U is $\mathcal{M}$.
  - Part ii tells us that $j^* \circ j_{!*} = \mathrm{Id}$, so clearly $j_{!*}(\mathcal{M})|_U \simeq \mathcal{M}$.
  - Suppose $\mathcal{N} \subset \mathcal{H}^0(j_*(\mathcal{M}))$ were a larger $\mathcal{D}_X$-submodule, still satisfying $\mathcal{N}|_U \simeq \mathcal{M}$.
  - Then the first assertion implies that
  $$j_{!*}(\mathcal{M}) = j_{!*}(\mathcal{N}|_U) \subseteq \mathcal{N} \subseteq \mathcal{H}^0(j_*(\mathcal{M})).$$
  This shows that $\mathcal{N} \supseteq j_{!*}(\mathcal{M})$ for all such $\mathcal{N}$, hence $j_{!*}(\mathcal{M})$ is the minimal such $\mathcal{N}$.
- The dual statement, that $j_{!*}(\mathcal{M})$ is the smallest quotient of $\mathcal{H}^0(j_!(\mathcal{M}))$ whose restriction to U is $\mathcal{M}$ follows from duality (applying $\mathbb{D}$).

(v)
- First we show that $j_{!*}(\mathcal{M})$ satisfies the conditions: that $j_{!*}(\mathcal{M})$ is an example of a quasicoherent $\mathcal{D}_X$-module whose restriction to U is $\mathcal{M}$, and has no submodule of quotient module supported on $X \setminus U$.
  - Clearly $j_{!*}(\mathcal{M})|_U = \mathcal{M}$.
  - Note that $\mathcal{H}^0(j_*(\mathcal{M})) = {}^\circ j_*(\mathcal{M})$, where ${}^\circ j_*$ is the underived quasicoherent $\mathcal{O}$-module pushforward.
  - Already we have $j_{!*}(\mathcal{M}) \subseteq \mathcal{H}^0(j_*(\mathcal{M})) = {}^\circ j_*(\mathcal{M})$, which has no $\mathcal{O}_X$-submodules supported on $X \setminus U$.
  - Duality implies that $j_{!*}(\mathcal{M})$ also has no quotient objects supported on $X \setminus U$.
  - Therefore $j_{!*}(\mathcal{M})$ works.
- Next we'll show it's the unique such one.
  - Suppose $\mathcal{N}$ also satisfies the conditions. In particular, $\mathcal{N}|_U \simeq \mathcal{M}$.
  - Since $\mathcal{N}$ has no subobjects supported on $X \setminus U$, then $\mathcal{N} \hookrightarrow \mathcal{H}^0(j_*(\mathcal{N}|_U))$.
  - Since $\mathcal{N}$ has no quotient objects supported on $X \setminus U$, then $\mathcal{H}^0(j_!(\mathcal{N}|_U)) \twoheadrightarrow \mathcal{N}$.
  - Recall that $\gamma_j(\mathcal{N}|_U)$ factors as the composition
  $$\gamma_j(\mathcal{N}|_U): \quad \mathcal{H}^0(j_!(\mathcal{N}|_U)) \twoheadrightarrow \mathcal{N} \hookrightarrow \mathcal{H}^0(j_*(\mathcal{N}|_U)).$$
  - Therefore
  $$j_{!*}(\mathcal{M}) = j_{!*}(\mathcal{N}|_U) = \operatorname{im}(\gamma_j(\mathcal{N}|_U)) = \mathcal{N}.$$

(vi)
- Once again, write $f = j \circ i$ as the composition of open and closed embeddings.
- When $f = i$ is a closed embedding, this is clear by Kashiwara's equivalence theorem (4.21).
- Now suppose $f = j$ is an open embedding.
  - Then $\mathcal{H}^0(j_*(\mathcal{M}))$ has a simple submodule $\mathcal{N} \hookrightarrow \mathcal{H}^0(j_*(\mathcal{M}))$.
  - Restrict this injection to U: we have $\mathcal{N}|_U \hookrightarrow \mathcal{H}^0(j_*(\mathcal{M}))|_U = \mathcal{M}$.
  - Since $\mathcal{M}$ is simple and $\mathcal{N}|_U \neq 0$, then $\mathcal{N}|_U = \mathcal{M}$.
  - By part iv, we have that $j_{!*}(\mathcal{M}) = j_{!*}(\mathcal{N}|_U) \subseteq \mathcal{N}$.

  - Therefore $j_{!*}(\mathcal{M}) = \mathcal{N}$, a simple submodule of $\mathcal{H}^0(j_*(\mathcal{M}))$. Since this works for every simple submodule, it follows that there is a unique such simple submodule, which coincides with $j_{!*}(\mathcal{M})$.
  - The statement about the unique simple quotient follows by duality.

(vii)
- Once again, write $f = j \circ i$.
- Then $i_{!*} = i_*$ is fully faithful by Kashiwara's equivalence (4.21).
- Assume $f = j$ is an open embedding now.
  - Since $j^* \circ j_{!*} \simeq \mathrm{Id}$. then $j_{!*}$ must be faithful (i.e., injective on hom-sets).
  - Now let's show fullness.
    - We need to show surjectivity of $\mathrm{Hom}(\mathcal{M}_1, \mathcal{M}_2) \to \mathrm{Hom}(j_{!*}(\mathcal{M}_1), j_{!*}(\mathcal{M}_2))$.
    - From the fact that $j^* \circ j_{!*} \simeq \mathrm{Id}$, we have a composition

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{D}_W}(\mathcal{M}_1, \mathcal{M}_2) \arrow[r, "j_{!*}"] \arrow[rr, bend left, "\mathrm{id}"] & \mathrm{Hom}_{\mathcal{D}_X}(j_{!*}(\mathcal{M}_1), j_{!*}(\mathcal{M}_2)) \arrow[r, "j^*"] & \mathrm{Hom}_{\mathcal{D}_W}(\mathcal{M}_1, \mathcal{M}_2). \end{tikzcd}$$

    - Thus surjectivity of the first map is equivalent to injectivity of the second map.
    - Now take $\varphi \in \mathrm{Hom}(j_{!*}(\mathcal{M}_1), j_{!*}(\mathcal{M}_2))$, such that $\varphi \mapsto 0$.
    - Then $j^*\varphi = 0$, i.e., $\mathrm{im}(\varphi)|_U = 0$, which implies that $\mathrm{im}(\varphi)$ is supported on $X \setminus U$.
    - But $\mathrm{im}(\varphi) \subset j_{!*}(\mathcal{M}_2)$, which by part v has no subobjects supported on $X \setminus U$. Therefore $\mathrm{im}(\varphi) = 0 \implies \varphi = 0$, proving the claim.

□

We also need to know:

Proposition 6.6: The intermediate extensions compose functorially. In other words, if $W \overset{g}{\hookrightarrow} V \overset{f}{\hookrightarrow} X$ are locally closed embeddings of smooth irreducible varieties, then there's a functorial isomorphism

$$(f \circ g)_{!*} \xrightarrow{\sim} f_{!*} \circ g_{!*}.$$

Proof, rough sketch. If both f, g are open embeddings, then we can use the properties in (6.4) to construct an isomorphism. If f, g are closed embeddings, then the intermediate extension is just the lower star (which is also lower shriek), which are t-exact functors. The main step is to write both f, g are a composition of open and closed embeddings, then rewriting the compositions so that the open and closed embeddings are adjacent, hence we can apply the previous cases. □

6.4 Classifying the simple holonomic $\mathcal{D}$-modules

One *very* useful reason to consider intermediate extensions is that it classifies the simple holonomic $\mathcal{D}$-modules.

Theorem 6.7: Let X be a smooth irreducible variety. Let $\mathcal{M}$ be a $\mathcal{D}_X$-module. Then $\mathcal{M}$ is simple and holonomic iff $\mathcal{M} \simeq f_{!*}(\mathcal{E})$, where $f : W \hookrightarrow X$ is a locally closed embedding of a smooth irreducible (sub)variety W into X, and $\mathcal{E}$ is a simple $\mathcal{D}_W$-module which is coherent as an $\mathcal{O}_W$-module.
In other words:

$$\{\text{Simple holonomic } \mathcal{D}_X\text{-modules}\} \iff \{f_{!*}\,(\text{simple, coherent as } \mathcal{O}_W\text{-module})\}\,.$$

Proof.

- One direction is easy. If $\mathcal{E}$ is a simple $\mathcal{D}_W$-module, then $f_{!*}(\mathcal{E})$ is a simple $\mathcal{D}_X$-module, by Theorem 6.4vi. (We don't even need that it's $\mathcal{O}_W$-coherent.)
- Suppose now that $\mathcal{M}$ is a simple $\mathcal{D}_X$-module. We will show that $\mathcal{M} \simeq f_{!*}(\mathcal{E})$, where $\mathcal{E}$ is a simple $\mathcal{D}_W$-module which is coherent as an $\mathcal{O}_W$-module.
  - First pick any open U such that $\mathcal{M}|_U \neq 0$. Equivalently, $\mathrm{supp}(\mathcal{M}) \cap U \neq$, or equivalently, $\Gamma_Z(\mathcal{M}) = 0$ (because $\mathcal{M}$ is simple, so long exact sequence in cohomology of the exact triangle associated to $U \subset X$

gives us the equivalence of these conditions). Let $j : U \hookrightarrow X$.

- Then Theorem 6.4iv tells us that $0 \neq j_{!*}(\mathcal{M}|_U) \subseteq \mathcal{M}$, where $\mathcal{M}$ is simple, so $j_{!*}(\mathcal{M}|_U) = \mathcal{M}$.
- Therefore $\mathcal{M}|_U$ is also simple (since $j_{!*}$ preserves injections, see Theorem 6.4iii).
- Now since $j_{!*}(\mathcal{M}|_U) \simeq \mathcal{M}$, and the intermediate extension functor composes well, we may replace X with any U such that $\mathcal{M}|_U \neq 0$.
- In particular, $\mathrm{supp}(\mathcal{M})$ is a closed subset of X which may not be smooth; but after replacing X by a suitable open set which avoids the singular points of $\mathrm{supp}(\mathcal{M})$, we may assume $\mathrm{supp}(\mathcal{M})$ is smooth and irreducible.
- Let $i : \mathrm{supp}(\mathcal{M}) \hookrightarrow X$ be the closed embedding (after our modifications).
- Then Kashiwara 4.21 implies that $\mathcal{M} \simeq i_*(\mathcal{N}) = i_{!*}(\mathcal{N})$ for some simple $\mathcal{D}_{\mathrm{supp}(\mathcal{M})}$-module $\mathcal{N}$.
- So we may assume that $X = \mathrm{supp}(\mathcal{M})$, having reduced to the case of $\mathcal{N}$, a $\mathcal{D}$-module on $\mathrm{supp}(\mathcal{M})$.
- If $\mathcal{M}$ is such that its support is the entirety of X, then we can always find some open $U \subset X$ such that $\mathcal{M}|_U$ is $\mathcal{O}_U$-coherent (see Theorem 5.3iii; apply to the open subset $X = Z_0 \setminus Z_1$).
- From the previous reduction, $\mathcal{M} \simeq j_{!*}(\mathcal{M}|_U)$, but now $\mathcal{M}|_U$ is an $\mathcal{O}_U$-coherent $\mathcal{D}_U$-module!
- Unraveling all of the reductions, we have our $\mathcal{E}$ the $\mathcal{O}_W$-coherent $\mathcal{D}$-module on W.
  W is obtained by first shrinking X to an open set where $\mathcal{M}|_U \neq 0$ and $U \cap \mathrm{supp}(\mathcal{M})$ is nonempty, smooth, and irreducible; then shrinking to $\mathrm{supp}(\mathcal{M})$ inside of this open set (here we switch $\mathcal{M}$ to some $\mathcal{N}$ by Kashiwara's equivalence); then finally shrinking this subvariety to an open set where $\mathcal{N}$ is $\mathcal{O}$-coherent.
  $\mathcal{E}$ is obtained by doing this process and keeping track of the $\mathcal{D}$-module. In the end, letting $f : W \hookrightarrow X$ be the locally closed embedding, then $\mathcal{M} \simeq f_{!*}(\mathcal{E})$, and $\mathcal{E}$ is $\mathcal{O}_W$-coherent.

□

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References

[HTT07] Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki. *D-modules, perverse sheaves, and representation theory*, volume 236. Springer Science & Business Media, 2007.

[Mus] Mircea Mustata. D-modules and singularities lecture notes for math 732, winter 2023.