# 18.747 Fall 2023

## Infinite-dimensional Lie algebras

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The professor for this course is Leonid Rybnikov. As always, all errors are my fault; please send me any that you find!

# Sep 6

Our goal in this course is to study most important infinite dimensional Lie algebras and their representations.

In the finite dim case, semisimple Lie algebras constitute the nicest and most important examples. This is done by highest weight representations, category $\mathcal{O}$, etc.

We are interested in infinite dimensional Lie algebras that behave similar to semisimple finite-dim Lie algebras, exhibiting such behavior as highest weight, etc.

We are mostly interested in:

- affine Kac-Moody Lie algebras
- Virasoro algebra
- Heisenberg algebra
- etc.

Our main references will be:

- Kac-Raina, *Bombay Lectures on Highest Weight Representations of Infinite DImensional Lie Algebras*
- Darij Grinberg's notes from 2012, taught by Pavel Etingof
- Victor Kac, *Infinite dimensional Lie algebras*
- the first chapter of D. B. Fuks, *Cohomology of Infinite-Dimensional Lie Algebras*

## Cohomology of Lie algebras

Let $\mathfrak{g}$ be a Lie algebra, $M$ be a $\mathfrak{g}$-module.
In general, we work over $\mathbb{C}$.

**Definition 1.**

We define $H^\bullet(\mathfrak{g}, M) := \mathrm{Ext}^\bullet_{\mathcal{U}(\mathfrak{g})}(\mathbb{1}, M)$ as a graded vector space, where $\mathbb{1}$ is the trivial $\mathfrak{g}$-module.

**Example 2.**

Let $\mathfrak{g}$ be an abelian Lie algebra, $\mathfrak{g} \cong \mathbb{C}^N$. How do we compute $H^\bullet$? We do this by taking projective resolution of $\mathbb{1}$ and taking Hom with $M$.
So we start with $0 \leftarrow \mathbb{1}$, then need a surjection from a free $\mathcal{U}(\mathfrak{g})$-module. Let's take $S\mathfrak{g}$, the symmetric algebra. We end up with an exact sequence

$$0 \leftarrow \mathbb{1} \leftarrow S\mathfrak{g} \leftarrow S\mathfrak{g} \otimes \mathfrak{g} \leftarrow S\mathfrak{g} \otimes \Lambda^2\mathfrak{g} \leftarrow S\mathfrak{g} \otimes \Lambda^3\mathfrak{g} \leftarrow \cdots,$$

this is called the **Koszul complex**. Now we must apply $\mathrm{Hom}(-, M)$ to this sequence. But

$$\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(S\mathfrak{g} \otimes \Lambda^k\mathfrak{g}, M) \cong M \otimes \Lambda^k\mathfrak{g}^* \cong \mathrm{Hom}_{\mathbb{C}}(\Lambda^k\mathfrak{g}, M).$$

So we get a complex

$$0 \to M \to \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, M) \to \mathrm{Hom}_{\mathbb{C}}(\Lambda^2\mathfrak{g}, M) \to \cdots$$

where if $\omega \in \mathrm{Hom}_{\mathbb{C}}(\Lambda^k\mathfrak{g}, M)$, then $d\omega(y_1, y_2, \dots, y_{k+1}) = \sum(-1)^{i-1} y_i \omega(y_1, \dots, \hat{y}_i, \dots, y_{k+1})$.

## General $\mathfrak{g}$.

We have the similar Koszul complex

$$0 \leftarrow \mathbb{C} \leftarrow \mathcal{U}(\mathfrak{g}) \leftarrow \mathcal{U}(\mathfrak{g}) \otimes \mathfrak{g} \leftarrow \mathcal{U}(\mathfrak{g}) \otimes \Lambda^2\mathfrak{g} \leftarrow \cdots,$$

and the differential maps

$$\begin{aligned} x \otimes y_1 \wedge \cdots \wedge y_k \mapsto & \sum(-1)^{i-1} x y_i \otimes y_1 \wedge \cdots \wedge \hat{y}_i \wedge \cdots \wedge y_k \\ & + \sum(-1)^{i+j-1} x \otimes [y_i, y_j] \wedge y_1 \wedge \dots \hat{y}_i \dots \hat{y}_j \dots y_k. \end{aligned}$$

Using the Jacobian identity, we can see that this is a differential, hence this is a complex.
The associated graded of this complex is the above example (using symmetric algebras), which is acyclic, hence this complex is acyclic. This implies the Chevalley complex.

**Corollary 3.**

We have the following exact complex, known as the **Chevalley complex**, which computes $H^\bullet(\mathfrak{g}, M)$:

$$M \to \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, M) \to \mathrm{Hom}_{\mathbb{C}}(\Lambda^2\mathfrak{g}, M) \to \dots$$

where if $\omega \in \mathrm{Hom}_{\mathbb{C}}(\Lambda^k \mathfrak{g}, M)$, then

$$\begin{aligned} d\omega(y_1, \ldots, y_{k+1}) = &\sum (-1)^{i-1} y_i \omega(y_1, \ldots, \hat{y}_i, \ldots, y_{k+1}) \\ &+ \sum_{i<j} (-1)^{i+j-1} \omega([y_i, y_j], y_1, \ldots, \hat{y}_i, \ldots, \hat{y}_j, \ldots, y_{k+1}). \end{aligned}$$

**Definition 4.**

$H^\bullet(\mathfrak{g}) := H^\bullet(\mathfrak{g}, \mathbb{1})$ with trivial coefficients.
It is computed by the Chevalley-Eilenberg complex

$$0 \to \mathbb{C} \to \mathfrak{g}^* \to \Lambda^2 \mathfrak{g}^* \to \Lambda^3 \mathfrak{g}^* \to \Lambda^4 \mathfrak{g}^* \to \cdots.$$

**Remark 5.**

This has some topological meaning if $\mathfrak{g} = T_e G$ for some Lie group $G$. Since it's a trivial representation, the first sum in $d\omega$ is zero, as it has the action of $y_i$ on it; thus the complex is computed by the complex $\Lambda^\bullet \mathfrak{g}^*$ with the differential given by the second sum in $d\omega$ above. But $\Lambda^\bullet \mathfrak{g}^* = \Omega(G)^{\text{left-right}}$, i.e. Left and right invariant differential forms on $G$, which are uniquely determined by their values at the unit $e \in G$. This is a subcomplex of the deRham complex $\Omega(G)$. If $G$ is compact (over $\mathbb{R}$), then we have a quasi-isomorphism $\Omega(G)^{\text{left-right}} \subset \Omega(G)$, hence $H^\bullet(G) = H^\bullet_{\mathrm{dR}}(G)$.

## Meaning of $H^0, H^1, H^2$.

- $H^0(\mathfrak{g}, M) = \mathrm{Hom}_{\mathfrak{g}}(\mathbb{1}, M) = M^{\mathfrak{g}}$, the space of $\mathfrak{g}$-invariants in $M$. Therefore $H^0(\mathfrak{g}) = \mathbb{C}$.
- $H^1(\mathfrak{g}, M) = \mathrm{Ext}^1(\mathbb{1}, M)$, so $H^1(\mathfrak{g}) = \mathrm{Hom}(\mathfrak{g}, \mathbb{1}) = \ker(\mathfrak{g}^* \to \Lambda^2 \mathfrak{g}^*) = (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$.
- Prop: $H^2(\mathfrak{g})$ are equivalent classes of central extensions $0 \to \mathbb{C} \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0$ where the first map is the embedding of the central subalgebra. The proof: we just compare this with the definition of $H^2$, which is $\ker(\Lambda^2 \mathfrak{g}^* \to \Lambda^3 \mathfrak{g}^*)/\mathrm{im}(\Lambda^1 \mathfrak{g}^* \to \Lambda^2 \mathfrak{g}^*)$. Note that for any central extension $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}c$, the commutator for $x, y \in \mathfrak{g} \subset \tilde{\mathfrak{g}}$ Becomes $[x, y]_{\tilde{\mathfrak{g}}} = [x, y]_{\mathfrak{g}} + c \cdot \omega(x, y)$ . Note that $\omega$ is clearly skew-symmetric (as the difference of two commutators) hence is in $\Lambda^2 \mathfrak{g}^*$, and furthermore the Jacobi identity implies that it is in the kernel of $\Lambda^2 \mathfrak{g}^* \to \Lambda^3 \mathfrak{g}^*$. Now the choice of splitting $\tilde{\mathfrak{g}} \cong \mathfrak{g} \oplus \mathbb{C}c$ corresponds to some $\nu \in \mathfrak{g}^*$, which changes $\omega$ by $d\nu = \nu([x, y])$: the choice of splitting in the short exact sequence $0 \to \mathbb{C}c \to \tilde{\mathfrak{g}} \to \mathfrak{g}$ is non-canonical and determined up to scalar multiples of $c$, which are precisely determined by maps $\mathfrak{g} \to \mathbb{C}c \cong \mathbb{C}$, which are just elements of $\mathfrak{g}^*$. (The coboundary condition is equivalent to the adjustment being a true lift $\mathfrak{g} \to \tilde{\mathfrak{g}}$.)

## Methods of computing $H^\bullet(\mathfrak{g}, M)$

**Theorem 6.**

The (Chevalley) complex $C^\bullet(\mathfrak{g}, M)$ is a $\mathfrak{g}$-module where the $\mathfrak{g}$-action commutes with the differential, hence $d$ is a $\mathfrak{g}$-endomorphism. This implies that $\mathfrak{g}$ acts on $H^\bullet(\mathfrak{g}, M)$, and in fact this action is trivial.

***Proof.***
Let $x \in \mathfrak{g}$, let its action be $L_x$. We have $L_x = d \circ \iota_x + \iota_x \circ d$, where $\iota_x$ is substitution of $x$; this follows from the formula for $d\omega$. Then for a cocycle $\omega$, $L_x(\omega) = d\iota_x(\omega)$ which is a coboundary, hence $0$ in $H^\bullet$. $\square$

**Corollary 7.**

Suppose $\mathfrak{g}$ is inner graded, i.e. giving by $\mathrm{ad}_x$ for $x \in \mathfrak{g}$. Then

$$C^\bullet(\mathfrak{g}) = \bigoplus_\lambda C^\bullet(\mathfrak{g})_\lambda$$

where $\lambda$ are eigenvalues of $\mathrm{ad}_x$. Then by the previous theorem, all the components where $\lambda \neq 0$ must be acyclic, so $H^\bullet(\mathfrak{g}) = H^\bullet(C^\bullet(\mathfrak{g})_0)$.

# Sep 8

Comment on the homework.

**Theorem 8.**

$\mathfrak{g}$ acts trivially on $H^\bullet(\mathfrak{g}, M)$.

**Corollary 9.**

Suppose $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}$, such that $\mathfrak{g}_i = \{x \in \mathfrak{g} \mid \mathrm{ad}_h x = ix\}$ for some $h \in \mathfrak{g}$ (i.e., this grading is **internal**). Then $H^\bullet(\mathfrak{g})$ is concentrated in degree $0$.

**Example 10.**

Suppose $\mathfrak{g}$ is finite dimensional and simple. Now consider a principal $\mathfrak{sl}_2 \subset \mathfrak{g}$, with elements $e, h, f$. Then $\mathrm{ad}_h : \mathfrak{g} \to \mathfrak{g}$ acts by integer eigenvalues. This makes $\mathfrak{g}$ a graded Lie algebra

$$\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n,$$

and since $h$ is a regular semisimple element, we have $h \in \mathfrak{g}_0$ (as well as $e \in \mathfrak{g}_2$ and $f \in \mathfrak{g}_{-2}$).
For example, consider $\mathfrak{g} = \mathfrak{gl}_n$. Then a principal $\mathfrak{sl}_2$ might consist of $e$ being the matrix of all 1s immediately above the diagonal, and zero elsewhere; $f$ the matrix of all 1s immediately below the diagonal and zero elsewhere; and $h$ the diagonal matrix with entries $(n-1, n-3, n-5, \dots, -(n-1))$. If we decompose $\mathfrak{gl}_n$ with the ad action of this $\mathfrak{sl}_2$. Then we have

$$\mathfrak{gl}_n = V_0 \oplus V_2 \oplus V_4 \oplus \cdots \oplus V_{2n-2},$$

where the dimensions are $1, 3, 5, \dots$.
In fact, you can view $\mathfrak{gl}_n$ in this way as a sum of irreducible representations of $\mathfrak{sl}_2$. The commutator operation can be written in terms of this decomposition where the coefficients are polynomials in $n$. Now if we view this $n$ as a parameter, we can then get an infinite-dimensional Lie algebra $\mathfrak{gl}_t$ for any $t \in \mathbb{C}$, which is a very interesting Lie algebra! For example, it is graded, and you can explicitly describe the graded components. When $t \in \mathbb{Z}_{\geq 0}$, this algebra has a nontrivial ideal, which you can quotient out by to get the usual $\mathfrak{gl}_n$. Analogously with Verma modules, they are generally irreducible, except at (certain) integer values upon which they have a finite dimensional irreducible quotient.
Another remark: for general $t$, $\mathfrak{gl}_t = \mathcal{U}(\mathfrak{sl}_2)/(C - \frac{(t+1)(t-1)}{2})$, and for nonnegative integer $t$, this has a finite-dimensional quotient.

**Example 11** (Homework).

Let $W = \mathrm{Der}\,(\mathbb{C}[z, z^{-1}]) = \mathrm{span}\{L_i = z^{i+1}\frac{\partial}{\partial z}\}$ be the Witt algebra. Then $\mathrm{ad}\ L_0 = \mathrm{ad}\ z\frac{\partial}{\partial z}$ gives a grading by the integers.

## More about Lie algebra cohomology.

Note that

$$H^\bullet(\mathfrak{g}, M) = \mathrm{Ext}^\bullet_{\mathcal{U}(\mathfrak{g})}(\mathbb{1}, M) \implies \mathrm{Ext}^\bullet_{\mathcal{U}(\mathfrak{g})}(N, M) = H^\bullet(\mathfrak{g}, \mathrm{Hom}_{\mathfrak{g}}(N, M)).$$

There are some particular meanings to cohomology groups of the adjoint representation (recall, they are also $\mathfrak{g}$-modules).
So $H^1(\mathfrak{g}, \mathfrak{g})$ are maps $\alpha : \mathfrak{g} \to \mathfrak{g}$ such that $\alpha([X, Y]) = [\alpha(X), Y] + [X, \alpha(Y)]$ modulo $\alpha = \mathrm{ad}_h$,

i.e. $H^1(\mathfrak{g},\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g})$, the derivations of $\mathfrak{g}$ modulo the inner derivations of $\mathfrak{g}$.
Also, $H^2(\mathfrak{g},\mathfrak{g})$ consists of infinitesimal deformations of $\mathfrak{g}$. More precisely, this is the space of all Lie structures $[,]$ on $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[\hbar]/\hbar^2$ which become the usual Lie bracket $[,]_{\mathfrak{g}}$ modulo $\hbar$ (this is the cocycle part); this is taken modulo the transformations which are identity modulo $\hbar$ (this is the coboundary).

## More about computations.

### Laplace operator.

**Definition 12** (Laplace operator)**.**

Consider $C^\bullet(\mathfrak{g}, M)$, and $\langle , \rangle$ a Hermitian inner product on $C^\bullet$. Then we can define $d^+$ the Hermitian conjugate of $d$. Now define $\Delta := dd^+ + d^+d$.

**Theorem 13.**

1. $\Delta$ commutes with $d$, $\Delta$ is self-adjoint and therefore diagonalizable, which implies that
$$C^\bullet(\mathfrak{g}, M) = \bigoplus_{\lambda \in \mathrm{Spec}\,\Delta} C^\bullet(\mathfrak{g}, M)_\lambda.$$
2. For $\lambda \neq 0$, $C^\bullet(\mathfrak{g}, M)_\lambda$ is acyclic, and $d|_{C^\bullet(\mathfrak{g},M)_0} = 0$, which implies that
$$H^\bullet(\mathfrak{g}, M) = C^\bullet(\mathfrak{g}, M)_0.$$
If $\mathbb{1}$ and is semisimple or compact Lie algebra, then $C^\bullet(\mathfrak{g}, M)_0$ is just the subcomplex of bi-invariant (i.e., left- and right- invariant) differential forms.

***Proof.***

1. is easy
2. If $\lambda \neq 0$, then if $\Delta\omega = \lambda\omega$, then $\omega = dd^+\omega/\lambda$. If $\lambda = 0$, then $0 = \langle \Delta\omega, \omega \rangle = \langle d\omega, d\omega \rangle + \langle d^+\omega, d^+\omega \rangle$, now using that it's positive definite, we obtain that they are both zero.

□

**Corollary 14.**

We can compute $H^\bullet(\mathfrak{g}, M)$ for any semisimple $\mathfrak{g}$ and finite-dimensional irreducible $M$: it is 0 if $M \neq \mathbb{1}$ since it's an Ext functor from trivial to nontrivial, and for $M = \mathbb{1}$, we have that $\Delta$ acts on

$\Lambda^\bullet\mathfrak{g}^*$ as the Casimir operator, so

$$(\Lambda^\bullet\mathfrak{g}^*)_0 = (\Lambda^\bullet\mathfrak{g}^*)^{\mathfrak{g}} = H^\bullet(\mathfrak{g}).$$

In particular, $H^1(\mathfrak{g}) = H^2(\mathfrak{g}) = 0$, so no characters, central extensions, as well as no deformations Additionally, $H^1_{dR}(G) = 0$ for any compact connected Lie group $G$. This implies that $H^1(G)$ is finite, hence $\pi_1(G)$ is finite, and thus there exists a simply connected cover $\widetilde{G}$ which crucially is still compact!

Furthermore, $H^3(\mathfrak{g}) \neq 0$: the reason is that $(\Lambda^3\mathfrak{g}^*)^{\mathfrak{g}} \neq 0$, as the commutator element is nonzero! Since $\mathfrak{g}$ is semisimple/reductive, the adjoint and coadjoint representations are isomorphic, so we identify $\mathfrak{g} \cong \mathfrak{g}^*$, hence any compact Lie group has a nonzero $H^3$.

As an additional corollary, there are no associative division algebras of dimension 8 over $\mathbb{R}$. The only parallelizable spheres are $S^1, S^3, S^7$. But why can't $S^7$ be a division algebra as well? Well, if it were, it would be a compact Lie group, and must have nonzero $H^3$, which isn't the case.

# Sep 11

Today, we aim to describe computation of:

1. $H^\bullet(\mathfrak{g})$ for $\mathfrak{g}$ a semisimple/reductive Lie algebra. Will also explain how these are related to generalized Chern classes.
2. $H^\bullet(\mathfrak{n})$ for $\mathfrak{n} \subset \mathfrak{g}$ a maximal nilpotent Lie subalgebra. Will also explain how these are related to Weyl character formulas.

Let's start with 1.

Suppose $\mathfrak{g}$ is semisimple. Recall that for an irreducible representation $M$, $H^\bullet(\mathfrak{g}, M)$ is nontrivial iff $M$ is trivial (reason is that $\mathrm{Ext}^\bullet_{\mathfrak{U}(\mathfrak{g})}(\mathbb{1}, M) = 0$). We can also use the Laplace operator approach, which acts by Casimir element.

Our goal is compute $H^\bullet(\mathfrak{g})$. This is interesting! We have two approaches: Laplace operator, and complex of bi-invariant differential forms. Then $H^\bullet(\mathfrak{g}) = \Lambda^\bullet(\mathfrak{g}^*)^{\mathfrak{g}}$ (invariants in the exterior power of the co-adjoint representation). For a semisimple/reductive Lie algebra, the adjoint and coadjoint representation are isomorphic $\mathfrak{g} \cong \mathfrak{g}^*$, hence $H^\bullet(\mathfrak{g}) \cong \Lambda^\bullet(\mathfrak{g})^{\mathfrak{g}}$.

## Question: What is $\Lambda^\bullet(\mathfrak{g}^*)^{\mathfrak{g}}$?

**Example 15.**

Then $\mathfrak{g} = \mathfrak{g}^* = \mathrm{Mat}_n$ as $\mathfrak{g}$-representations. Then we are considering $\Lambda^\bullet(\mathrm{Mat}_n)^{\mathsf{GL}_n}$. The appropriate version of the fundamental theorem of invariants says that this is a wedge algebra generated by

$M \mapsto \text{tr } M^k$ for powers of $k$. Furthermore, observe that $\text{tr } M^{2k} = 0$ for all even integers $2k$. Note that $\text{tr } M^{2k} = \text{tr } M \cdot M^{2k-1} = -\text{tr } M^{2k-1} \cdot M = -\text{tr } M^{2k} = 0$ since $M, M^{2k-1}$ are odd powers and the entries thus anti-commute.

As an example, consider $M = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$. Then $M^2 = \begin{pmatrix} x_{11}^2 + x_{12}x_{21} & x_{11}x_{12} + x_{12}x_{22} \\ x_{21}x_{11} + x_{22}x_{21} & x_{21}x_{12} + x_{22}^2 \end{pmatrix}$.

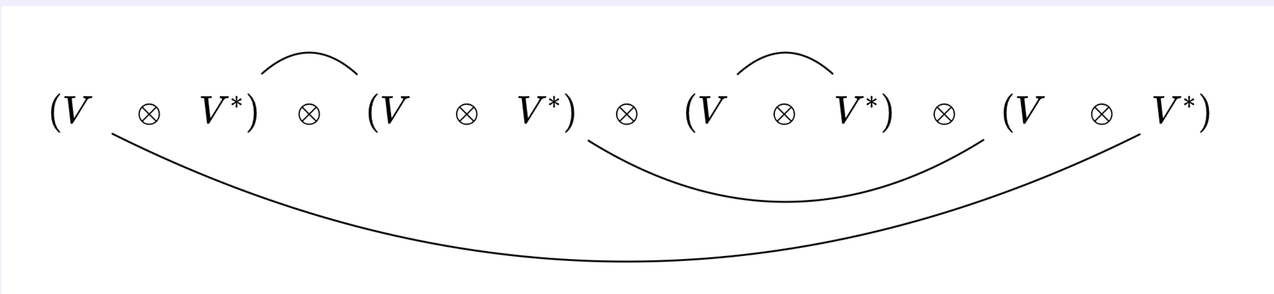
But $x_{11}^2 = x_{11} \wedge x_{11} = 0$, similarly $x_{22}^2 = 0$ (living in the exterior algebra), hence
$\text{tr } M^2 = x_{12} \wedge x_{21} + x_{21} \wedge x_{12} = 0$.

As a similar exercise, $\text{tr } M^3 \neq 0$.

To see why these elements generate $\Lambda^\bullet(\text{Mat}_n)^{\mathsf{GL}_n}$, note that $\text{Mat}_n = V \otimes V^*$ for $V = \mathbb{C}^n$. Now $\Lambda^\bullet(\text{Mat}_n)$ is a quotient of the tensor algebra
$T^\bullet(\text{Mat}_n) = \mathbb{C} \oplus V \otimes V^* \oplus V \otimes V^* \otimes V \otimes V^* \oplus \cdots$. Now the fundamental theorem of invariant theory says that the invariants of $(V \otimes V^*) \otimes (V \otimes V^*)$ are given by pairing the inner two entries and the outer two entries.

To see why, consider $(V \otimes V^*) \otimes (V \otimes V^*) \otimes (V \otimes V^*)$.

Suppose they are matched up like this.




Because one of them is internally paired (the third terms), this expression becomes $\text{tr } M^3 \cdot \text{tr } M$. So the decomposition into the product of traces corresponds to the cycle type of pairings of the tensor product.

The conclusion: $T_{2k-1}$ for $k = 1, 2, 3, \ldots$ generate $\Lambda(\text{Mat}_n)^{\mathsf{GL}_n}$. Since this is an infinite number of generators, this is dependent; so we can restrict to a finite number of generators. What is the cutoff?

**Theorem 16.**

$T_{2k-1}$ for $k = 1, 2, \ldots, n$ generate $\Lambda(\text{Mat}_n)^{\mathsf{GL}_n}$. Furthermore, there are no relations.

***Proof.***

We want to show that $T_{2k-1} = 0$ for all $k > n$.

Lemma: $M^{2n} = 0$. This is because $M^2$ is a $n \times n$ matrix with commutative coefficients, so we can apply Cayley-Hamilton theorem. Then the characteristic polynomial is a polynomial whose (non-leading) coefficients are homogeneous expressions of $\text{tr } M^{2k} = 0$, hence the characteristic polynomial is $t^n$, thus $(M^2)^n = 0$. (Note that this is equivalent to Amitsur-Levitski

theorem.)
The part where there are no relations is below. □

Therefore we can compute this algebra. The conclusion:

**Theorem 17.**

$H^\bullet(\mathfrak{gl}_n) = \Lambda^\bullet(T_1, T_3, \dots, T_{2n-1})$.

***Proof.***
Note that $H^{n^2}(\mathfrak{gl}_n) \neq 0$ (since it's equal to $H^{n^2}(U_n, \mathbb{C})$ where $U_n$ is a compact $n^2$-dimensional manifold), but there is only way to obtain something of degree $n^2$ from these generators, which is $T_1 \wedge T_3 \wedge \cdots \wedge T_{2n-1}$, hence must be nonzero, hence cannot have any relations (this wedge is contained in any nontrivial ideal in $\Lambda^\bullet(T_1, \dots, T_{2n-1})$. □

**Theorem 18.**

The cohomology ring always looks like the above case: $H^\bullet(\mathfrak{g}) = \Lambda(T_1 \dots, T_\ell)$ where $\deg T_i = 2m_i - 1$, $m_i$ are the exponents of $\mathfrak{g}$, and there are no relations amongst the $T_i$.

***Proof.***
Observe that $H^\bullet(\mathfrak{g})$ is a Hopf algebra, because $H^\bullet(\mathfrak{g}) = H^\bullet(G, \mathbb{C})$, and we have map $G \times G \to G$ (multiplication), hence obtain comultiplication map $\Delta : H^\bullet(G, \mathbb{C}) \to H^\bullet(G \times G, \mathbb{C})$, as well as a counit map obtained by the map $G \to \{\mathrm{pt}\}$ inducing $H^\bullet(G, \mathbb{C}) \to \mathbb{C}$.
It follows that $H^\bullet(\mathfrak{g})$ is supercommutative graded Hopf algebra with $H^0(\mathfrak{g}) = \mathbb{C}$. Therefore we can apply the appropriate version of the Milnor-Moore theorem, which implies that $H^\bullet(\mathfrak{g})$ is a free supercommutative algebra in some generators; since it's finite-dimensional, they must necessarily be odd degree, hence it is the free exterior power of odd degree generators. □

# Sep 13

Let $\mathfrak{g}$ be a semisimple reductive Lie algebra. Then $H^*(\mathfrak{g}) = \Lambda(\mathfrak{g}^*)^{\mathfrak{g}}$.

**Theorem 19.**

$H^\bullet(\mathfrak{g})$ is a free exterior algebra of some odd-degree generators $T_1, \dots, T_\ell$ where $\deg T_i = 2m_i + 1$ where $m_i$ are the exponents of $\mathfrak{g}$.

**Remark 20.**

$H^\bullet(\mathfrak{g}) = H^\bullet_{dR}(G, \mathbb{C})$, where $G$ is a compact group.

***Proof.***
It will follow from the proposition below, due to the fact that $H^\bullet(\mathfrak{g}) = H^\bullet_{dR}(G, \mathbb{C})$, where $G$ is a compact group, and the latter is a graded Hopf algebra, obtaining comultiplication from $G \times G \xrightarrow{\text{mult}} G$ and counit from $G \to \{\text{pt}\}$.

$\square$

**Proposition 21** (Weaker version of Milnor-Moore)**.**

Let $H$ be a graded Hopf algebra $H = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} H^i$, where each $H^i$ is finite dimensional, $H$ is supercommutative, and $H^0 = \mathbb{C}$. Then $H$ is a free supercommutative algebra in some homogeneous generators.

***Proof.***
First observe that the counit map $\varepsilon : H \to \mathbb{C}$ has the following form. Since $H$ is graded, $\varepsilon|_{H^{>0}} = 0$, it must annihilate everything of positive degree. Let $x$ have degree $n$. This implies that $\Delta(x) = x \otimes 1 + 1 \otimes x + \sum x_i(1) \otimes x_i^{(2)}$ where $0 < \deg x_i < n$.

Let $x_1, x_2, \dots, x_N$ be a minimal set of homogeneous generators, $\deg x_1 \leq \deg x_2 \leq \dots$. Now denote $H_n := \mathbb{C}\langle x_1, \dots, x_n \rangle$, so that $H_1 \subseteq H_2 \subseteq \dots$. To show that this is a free supercommutative algebra, we need to show that $H_{n-1} \otimes \mathbb{C}[x_n] \to H_n$ is an isomorphism (by $\mathbb{C}[x_n]$, I mean the polynomial algebra if $\deg x_n$ is even, and two dimensional exterior algebra if $\deg x_n$ is odd). Note that $H_n$ is closed with respect to $\Delta$.

Let $I \subset H_n$ be the ideal generated by $x_1, \dots, x_{n-1}, x_n^2$. Now consider the map

$$\Delta' := H_n \xrightarrow{\Delta} H_n \otimes H_n \xrightarrow{\text{id} \otimes -/I} H_n \otimes \mathbb{C}[x_n]/x_n^2.$$

Suppose there exists a nontrivial relation $\sum_{i=1}^k \alpha_i(x_1, x_2, \dots, x_{n-1}) x_n^i = 0$, of minimal degree. We can use $\Delta'$ to get a lower degree relation

$$H_n \otimes \mathbb{C}[x_n]/x_n^2 \ni 0 = \Delta'(0) = \sum_{i=1}^k \alpha_i \otimes 1 (x_n \otimes 1 + 1 \otimes x_n)^i = \sum i \alpha_i x_n^{i-1} \otimes x_i,$$

$$\implies \sum i \alpha_i (x_1, \dots, x_{n-1}) x_n^{i-1} = 0,$$

hence we obtain a relation of lower degree, contradiction.

$\square$

## About $\deg T_i$

We can relate $T_i$ to the generators of $S(\mathfrak{g}^*)^{\mathfrak{g}} = S(\mathfrak{h})^W = \mathbb{C}[P_1, P_2, \ldots, P_\ell]$ where $\deg P_i = m_i + 1$, the Weyl group invariants of the symmetric algebra of the Cartan subalgebra.
Consider the Weil algebra, a differential-graded supercommutative algebra:

$$W(\mathfrak{g}) := (\Lambda^\bullet(\mathfrak{g}^*) \otimes S^\bullet(\mathfrak{g}^*), \quad d)$$

where the degree of the $\mathfrak{g}^*$ in the wedge product is $1$, and the degree of the $\mathfrak{g}^*$ in the symmetric algebra is $2$. The differential (which satisfies the super Leibniz rule) looks as follows.

$$\begin{tikzcd}
\vdots \\
\bigwedge^2 \mathfrak{g}^* & 0 & \bigwedge^2 \mathfrak{g}^* \otimes \mathfrak{g}^* & \cdots \\
\mathfrak{g}^* \arrow[u] \arrow[drr, "\sim"] & 0 & \mathfrak{g}^* \otimes \mathfrak{g}^* & 0 & \mathfrak{g}^* \otimes S^2\mathfrak{g}^* \\
\mathbb{C} & 0 & \mathfrak{g}^* \arrow[u] & 0 & S^2\mathfrak{g}^* & \cdots
\end{tikzcd}$$

It suffices to define the differential $d$ on the generators, i.e. the two copies of $\mathfrak{g}^*$. On the exterior algebra copy, i.e. the left-most column, it is the sum of two maps: one is the map $\mathfrak{g}^* \to \Lambda^2\mathfrak{g}^*$ from the Chevalley complex (in green above), and the other is the isomorphism $\mathfrak{g}^* \to \mathfrak{g}^*$ (in red above). On the symmetric algebra copy, i.e. the middle column, the differential is just the blue map, sending $\mathfrak{g}^* \to \Lambda^2\mathfrak{g}^* \hookrightarrow \mathfrak{g}^* \otimes \mathfrak{g}^*$ (again using the Chevalley complex map).
One can check that the square of this differential is zero. As a complex, this Weil algebra is acyclic, because it is a free supercommutative algebra generated by degree $1$ space and its differential, which is isomorphic; it is free object of this sort, hence it is acyclic. You can also use a spectral sequence to compute the cohomology of $W(\mathfrak{g})$. On the $E^2$ page, we get $H^\bullet(\mathfrak{g}, S(\mathfrak{g})) = \Lambda(\mathfrak{g}^*)^{\mathfrak{g}} \otimes S(\mathfrak{g}^*)^{\mathfrak{g}}$.

## Cohomology of $\mathfrak{n}_-$.

Let $\mathfrak{n}_- \subset \mathfrak{g}$ be a maximal nilpotent subalgebra (since all of the maximal nilpotent subalgebras are conjugate, we may choose it to be $\mathfrak{n}_-$). Then

**Theorem 22.**

$H^\bullet(\mathfrak{n}_-) = \bigoplus_{w \in W} \mathbb{C}\xi_w$, where $\deg \xi_w = \ell(w)$.

***Proof.***
Here's a sketch of the proof. We have the BGG resolution of the trivial $\mathfrak{n}_-$-module:

$$\mathbb{C} \leftarrow M(0) \leftarrow \bigoplus M(s_i \cdot 0) \leftarrow \cdots \leftarrow \bigoplus_{\ell(w)=n} M(w \cdot 0).$$

Each $M(w \cdot 0)$ is a free $\mathfrak{n}_-$-module.

□

**Remark 23.**

This is not really an honest proof, as this uses "too much" information. In fact, it is possible to compute the cohomology directly using the Laplace operator, which is done in a paper by Kostant. (This was done before BGG!) He observed that you can deduce the Weyl character formula from his method, and moreover his method is general enough that you can also use it on infinite-dimensional Lie algebras, not just finite-dimensional ones.

### Examples of infinite-dimensional Lie algebras, and their representations.

$\mathfrak{gl}_\infty$ is some Lie algebra of operators in $\mathbb{C}^{\aleph_0} = \mathbb{C}[z, z^{-1}] = V$.
Want:

- operators of multiplication by $z^i$ - abelian Lie subalgebra
- differential operators, in particular $W = \mathrm{span}\{z^{i+1}\frac{\partial}{\partial z}\}_{i \in \mathbb{Z}}$ (these look like infinite matrices which is zero on all but finitely many diagonals)
- Generalized Jacobi matrices
  Fundamental representations (representations with highest weight, whose value on only simple root is 1, and 0 elsewhere). In the finite-dimensional case, these come from $\Lambda^k$. Therefore in the infinite-dimensional case these should look "something like" $\Lambda^{\infty/2}V := \mathrm{span}\{v_{i_1} \wedge v_{i_2} \wedge \ldots\}_{I=\{i_1<i_2<\ldots\}} \mid |I \setminus \mathbb{Z}_{\geq 0}| = |\mathbb{Z}_{\geq 0} \setminus I| < \infty$.

# Sep 15

# Legitimizing $\Lambda^{\frac{\infty}{2}}V$

### Recollection on spinor representations

Let's consider a simple Lie algebra $\mathfrak{gl}_n$. Then all fundamental representations are just $\Lambda^k\mathbb{C}^n$ where $\mathbb{C}^n$ denotes the tautological representation.
If we consider the fundamental representations of $\mathfrak{so}_n$, then we do indeed have the fundamental representations $\mathbb{C}^n$, $\Lambda^2\mathbb{C}^n$, etc. However, this is not all of them! There are one or two additional

fundamental representations which are not wedge powers of the tautological representation. The reason for this is that the Lie group $\mathsf{SO}_n$ is not simply connected, in particular $|\mathsf{SO}_n| \in \{2, 4\}$ , therefore there should be fundamental representations of $\widetilde{\mathsf{SO}_n}$ that do not factor through $\mathsf{SO}_n$. In fact, $\widetilde{\mathsf{SO}_n} \to \mathsf{SO}_n$ is a central extension, so we want representations of a central extension of $\mathsf{SO}_n$.
Where do these representations come from?
Let $U = \mathbb{C}^n$ be a vector space with symmetric bilinear form $B$ (non-degenerate).

**Definition 24.**

We denote the **Clifford algebra** $C\ell(U) := T(U)/(u_1u_2 + u_2u_1 - B(u_1, u_2) \mid u_1, u_2 \in U)$ to be a quotient of the tensor algebra.

It is an odd analog of the Weyl algebra, attached to a symplectic vector space. If $(V, \omega)$ is a symplectic vector space, then the Weyl algebra is $W(V) = T(V)/(v_1v_2 - v_2v_1 - \omega(v_1, v_2))$.
It has a natural filtration by assigning degree $1$ to all of the generators. Then the associated graded of the Clifford algebra is just the exterior algebra of $U$, i.e.

$$\operatorname{gr} C\ell(U) = \Lambda(U).$$

Now suppose that $V = X \oplus X^*$, where $X$ is some vector space (this is always the case when $V$ is an even-dimensional symplectic vector space, as you can choose a Lagrangian subspace), then $W(V) \cong \mathcal{D}(X)$, differential operators on this space.
Similarly to $W(V)$, if $U$ is even-dimensional, then (over $\mathbb{C}$) we can always always choose a Lagrangian subspace and a complement of a Lagrangian subspace, so we can write $U = V \oplus V^*$ (where $V, V^*$ are maximal isotropic) so that $B(v, v) = 0 = B(v^\vee, v^\vee)$ and $B(v, v^\vee) = v^\vee(v)$ for $v \in V$, $v^\vee \in V^*$. Then

$$C\ell(V \oplus V^*) \cong \mathcal{D}(\Lambda^\bullet(V)),$$

the algebra of super-differential operators. To any $v \in V$, we may send

$$V \ni v \mapsto (x \mapsto v \wedge x) : \Lambda^k V \to \Lambda^{k+1} V,$$

$$V^* \ni v^\vee \mapsto \partial_{v^\vee} : \Lambda^k V \to \Lambda^{k-1} V.$$

Let's compute this very explicitly. Let $x_1, \ldots, x_m$ be a basis of $V$.
Let $x_1^\vee, \ldots, x_m^\vee$ be the dual basis of $V^*$.
The isomorphism $C\ell(V \oplus V^*) \cong \mathcal{D}(\Lambda^\bullet(V))$ can be realized explicitly by specifying the images of the $x_i$ and the $x_j^\vee$. For the $x_i$, we map $x_i \mapsto (x_i \wedge -)$. For the $x_i^*$, we map it to $\partial_{x_i^\vee}$. But what is the action of $\partial_{x_i^\vee}$? It acts on a monomial $x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_n}$ as follows. If none of the $x_{i_j}$ are $x_i$, then it sends this monomial to $0$. If one of the $i_j$ indeed equals $i$, then we permute the vectors to make it the first entry (changing sign accordingly), then delete it. For example,

$$\partial_{x_1^\vee}(x_2 \wedge x_1) = -\partial_{x_1^\vee} x_1 \wedge x_2 = -x_2.$$

This action is indeed well-defined and is canonical, i.e. independent of the chosen basis. It is an easy exercise that the images of $x_i$ and $x_j^\vee$ indeed commute in $C\ell$.

**Proposition 25.**

$C\ell(V \oplus V^*) \cong \mathrm{End}(\Lambda(V))$.

**Remark 26.**

If $\dim U$ is odd, then $\dim C\ell(U) = 2^{\dim U}$, which is not a square and thus is not a matrix algebra, but it happens to be the direct sum of two matrix algebras.

***Proof.***
Here's a sketch of a proof. By using the images of $x_j^\vee$, we can take any nonzero element of $\Lambda(V)$ to $1 \in \Lambda(V)$ by killing the vectors one by one until they are reduced to $1$. Then clearly we can produce any element of $\Lambda(V)$ from $1$ by wedging with the appropriate elements, using the images of $x_i$.

□

## Constructing spinor representations

In the case of $U = V \oplus V^*$, we have that $C\ell(V \oplus V^*) \cong \mathrm{End}(\Lambda^\bullet(V))$, and there is a unique simple module (because it is a matrix algebra, thus the only simple is $\Lambda^\bullet(V)$)).
It is acted on by the orthogonal group:

$$\mathsf{SO}(U) \to \mathrm{Aut}(U).$$

On the other hand, $C\ell(U)$ is a matrix algebra, so all automorphisms are inner automorphisms. This implies that we may define a map $\mathsf{SO}(U) \to \mathsf{GL}(\Lambda^\bullet(V))$. This is a representation… right? NO!!! Not really. It is only a projective representation, as scalars (more precisely, the center) act by identity during conjugation. So in fact we have the following commutative square:

$$\begin{tikzcd} \widetilde{\mathsf{SO}(U)} \arrow[r] \arrow[d] & \mathsf{SO}(U) \arrow[d] \\ \mathsf{GL}(\Lambda^\bullet(V)) \arrow[r] & \mathsf{PGL}(\Lambda^\bullet(V)) \end{tikzcd}.$$

This representation does not necessarily factor through $\mathsf{SO}(U)$; thus, $\Lambda^\bullet(V)$ is a representation of $\widetilde{\mathsf{SO}(U)}$.
Now we take the differential of this representation to obtain the corresponding representation of the Lie algebra (note that on the level of Lie algebras, the central extension is trivial),

$$\mathfrak{so}(U) \to C\ell(U).$$

On the level of Lie algebras, we can describe this homomorphism explicitly. We have that $\mathfrak{so}(U) \cong \Lambda^2(U)$. On the other hand, $C\ell(U) \supset \Lambda^2(U) = \{u_1u_2 - u_2u_1 \mid u_1, u_2 \in U\}$, so the map of Lie algebras is just the composite

$$\mathfrak{so}(U) \xrightarrow{\sim} \Lambda^2(U) \hookrightarrow C\ell(U).$$

It follows that $\mathfrak{so}(U)$ is just the space of quadratic elements of the Clifford algebra $C\ell(U)$.
In particular if $U = V \oplus V^*$, then we have a copy of $\mathfrak{gl}(V) \subset C\ell(U)$ generated by elements $x_i x_j^\vee$ (or, more precisely, $x_i x_j^\vee - x_j^\vee x_i$). This copy of $\mathfrak{gl}(V)$ is contained in the copy of $\mathfrak{so}(U)$, but $\mathfrak{so}(U)$ has additional generators, $x_i x_j - x_j x_i$ and $x_i^\vee x_j^\vee - x_j^\vee x_i^\vee$. (Recall that the standard presentation of the form on $\mathfrak{so}(U)$ is $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$.)

## Semiifinite wedge spaces

It's an infinite-dimensional analogue of this construction.
Consider $V = \mathbb{C}[z, z^{-1}]$ and $V^*$. Now we consider $V$ as a graded space with $\deg z^i = i$. We may consider $V^*$ as the restricted dual of $V$, i.e. the direct sum of the duals of the graded components of $V$. Then we have the natural identification

$$V^* \cong \mathbb{C}[z, z^{-1}]\frac{\mathrm{d}z}{z}, \quad \langle f, g\rangle = \mathrm{Res}_{z=0} g(f) \quad \text{(i.e., coefficient of } \frac{1}{z}).$$

We can now consider $C\ell(V \oplus V^*)$.

**Definition 27** (Clifford algebra).

Let $V = \mathbb{C}[z, z^{-1}]$ and $V^*$ its restricted dual as above. Define

$$\begin{aligned} C\ell(V \oplus V^*) &:= \mathbb{C}\langle \psi_i, \psi_i^* \mid i \in \mathbb{Z}\rangle / \mathcal{I}, \\ \mathcal{I} &= \left\{\psi_i + \psi_j + \psi_j\psi_i = 0, \quad \psi_i^*\psi_j^* + \psi_j^*\psi_i^* = 0, \quad \psi_i\psi_j^* + \psi_j^*\psi_i = \delta_{i+j=0}\right\}. \end{aligned}$$

Here, we identify $\psi_i \leftrightarrow z^i$ and $\psi_i^* \leftrightarrow z^{i-1}\,\mathrm{d}z$.

Now, we want an irreducible representation of $C\ell(V \oplus V^*)$ containing $\Psi := \psi_0 \wedge \psi_1 \wedge \psi_2 \wedge \dots$. What properties should $\Psi$ satisfy? Well, we want $\psi_i \wedge \Psi = 0$ for $i \geq 0$ and $\psi_i^*\Psi = 0$ for $i > 0$ (because $\psi_i^* = \partial_{\psi_{-i}}$).

Now let us write

$$V \oplus V^* = \underbrace{V_+}_{\mathbb{C}[z] \oplus \mathbb{C}[z]\mathrm{d}z} \oplus \underbrace{V_-}_{z^{-1}\mathbb{C}[z^{-1}] \oplus z^{-1}\mathbb{C}[z^{-1}]\mathrm{d}z}.$$

These are both isotropic subspaces with respect to the bilinear form above. Therefore within the Clifford algebra, we have

$$C\ell(V\oplus V^*)\supset \Lambda(V_+),\Lambda(V_-).$$

Now consider $C\ell(V\oplus V^*)\otimes_{\Lambda(V_+)}\mathbb{1}$. This is the biggest cyclic representation of $C\ell$ containing $\Psi$, i.e. the universal representation containing $\Psi$.

Some facts which will be proved next time:

$$C\ell(V\oplus V^*)\otimes_{\Lambda(V_+)}\mathbb{1}=\bigoplus_{k=-\infty}^{\infty}\Lambda^{\frac{\infty}{2}+k}.$$

Just as how we have a copy of $\mathfrak{gl}(V)\subset C\ell(V\oplus V^*)$, we have a copy of $\mathfrak{gl}_\infty$ (generalized Jacobi matrices, consisting of zeros on all but finitely many diagonals) which can be embedded into a completion of $C\ell(V\oplus V^*)\curvearrowright\Lambda^{\frac{\infty}{2}+\bullet}$ .

# Sep 18

Today we will continue discussion of $\mathcal{F}:=\Lambda^{\frac{\infty}{2}+\bullet}$.

## Sources of central extensions

**Example 28.**

Let's look at finite-dimensional spinor representations. Let $U$ be an orthogonal vector space, and $\mathcal{B}:U\times U\to\mathbb{C}$ the corresponding symmetric bilinear form.
Then $C\ell(U)$ is naturally filtered by the degree of the generators (i.e. $\deg x=1$ for all $x\in U$). It is not graded because the supercommutator from two elements of $U$ can be a nonzero constant, but it is still $\mathbb{Z}/2\mathbb{Z}$-graded because the relations in $C\ell$ contains only terms of even degree. Therefore we have a canonical splitting

$$C\ell(U)=C\ell(U)_{\text{even}}\oplus C\ell(U)_{\text{odd}}.$$

Inside the even part, we can consider the subspace $C\ell(U)^{\le 2}_{\text{even}}$ of elements of degree $\le 2$. This is a nice subspace, because it is a Lie subalgebra with respect to the commutator. Indeed, the commutator of any two elements in this subspace is even and degree is strictly less than 4, hence must lie in this subspace. Moreover, we have a homomorphism

$$\begin{gathered}C\ell(U)^{\le 2}_{\text{even}}\xrightarrow{\text{ad}}\text{End}_{\mathbb{C}}(U),\\ y\mapsto[y,-]\text{ on }U\subset C\ell(U)^{\le 1}_{\text{odd}}.\end{gathered}$$

(Note that the commutator with $U$ lands in $C\ell(U)^{<3}_{\text{odd}}$.)
In fact, the image of $\text{ad}$ lands in $\mathfrak{so}(U,\mathcal{B})$: using the relation in $C\ell$, we have that

$$\begin{aligned}\mathcal{B}(\mathrm{ad}_y x_1, x_2) + \mathcal{B}(x_1, \mathrm{ad}_y x_2) &= (\mathrm{ad}_y x_1)x_2 + x_2 \mathrm{ad}_y x_1 + x_1 \mathrm{ad}_y x_2 + (\mathrm{ad}_y x_2)x_1, \\ &= [y, x_1x_2 + x_2x_1], \\ &= [y, \mathcal{B}(x_1, x_2)] = 0,\end{aligned}$$

since $\mathcal{B}(x_1, x_2) \in \mathbb{C}$.
So we have a homomorphism

$$C\ell(U, \mathcal{B}) \xrightarrow{\mathrm{ad}} \mathfrak{so}(U, \mathcal{B}),$$

which annihilates only constants, i.e. $\ker \mathrm{ad} = \mathbb{C} = C\ell^{\leq 0}$.

Let $\varphi : \mathfrak{so}(U, \mathcal{B}) \to C\ell(U)^{\leq 2}_{\text{even}}$ be a splitting. Let $F$ be an irreducible representation of $C\ell(U)$. Then $\varphi$ makes $F$ a representation of $\mathfrak{so}(U, \mathcal{B})$, as we have a composition of maps $\mathfrak{so}(U, \mathcal{B}) \to C\ell(U)^{\leq 2}_{\text{even}} \to \mathfrak{gl}(F)$, up to an additive constant, hence $F$ is a representation of some central extension of $\mathfrak{so}(U, \mathcal{B})$. But we have already constructed a central extension:

$$0 \to \mathbb{C} \to C\ell^{\leq 2}_{\text{even}} \xrightarrow{\mathrm{ad}} \mathfrak{so}(U, \mathcal{B}) \to 0,$$

where $\mathrm{ad}$ splits the injection $\mathfrak{so}(U, \mathcal{B}) \xrightarrow{\sim} C\ell(U, \mathcal{B})^{=2} \hookrightarrow C\ell(U, \mathcal{B})$.

Now we know that the composite $\mathfrak{so}(U, \mathcal{B}) \xrightarrow{\varphi} C\ell(U) \xrightarrow{\mathrm{ad}} \mathrm{End}_{\mathbb{C}}(C\ell(U))$ is a homomorphism. Suppose $F$ is a(n irreducible) representation of $C\ell(U)$, with mapping $\pi : C\ell(U) \to \mathrm{End}_{\mathbb{C}}(F)$. Then $\pi \circ \varphi : \mathfrak{so}(U, \mathcal{B}) \to \mathrm{End}(F)$ turns $\varphi$ into a representation of $F$… up to adding some endomorphism (of the representation):

$$(\pi \circ \varphi)([y_1, y_2]) = [\pi \circ \varphi(y_1), \pi \circ \varphi(y_2)] + z.$$

This $z$ term is something which commutes with everything in the Clifford algebra. If $F$ is irreducible, then by the Schur lemma, $z \in \mathbb{C} \cdot \mathrm{Id}$, hence some central extension of $\mathfrak{so}(U, \mathcal{B})$ acts on $F$. (In general, we only require a map such that the composition is a homomorphism, but the two defining maps do not need to be.)

## Generalization to $C\ell(\mathbb{C}[z, z^{-1}] \oplus \mathbb{C}[z, z^{-1}]\, \mathrm{d}z) \xleftarrow{\varphi} \mathfrak{gl}_\infty$

First, a technical point: Schur lemma requires vector spaces to be finite-dimensional.

**Lemma 29** (Schur lemma, infinite-dimensional analogue)**.**

Let $A$ be an associative $\mathbb{C}$-algebra, $M$ a simple $A$-module of countable dimension. Then $\mathrm{End}_A(M) = \mathbb{C}$.

***Proof.***
First, $\mathrm{End}_A(M)$ is a division algebra, since $M$ is simple.

Second, $\dim \operatorname{End}_A(M)$ is countable since $M$ is cyclic, hence if $M = A \cdot m$, then any $\varphi \in \operatorname{End}_A(M)$ is determined by $\varphi(m) \in M$, which has countable dimension.
Third, for any nonconstant $z \in \operatorname{End}_A(M)$, $z$ must be transcendental over $\mathbb{C}$ (because there are no finite extensions of $\mathbb{C}$, thus $z$ cannot satisfy a polynomial identity). Suppose for the sake of contradiction that $\mathbb{C}(z) \subset \operatorname{End}_A(M)$. But $\mathbb{C}(z)$ is already of uncountable dimension: $\left\{\frac{1}{z-c} \mid c \in \mathbb{C}\right\}$ are linearly independent, else there would be a polynomial relation, hence contradicting that $z$ must be transcendental. It follows that there exist no nonconstant functions in $\operatorname{End}_A(M)$.

□

# $C\ell(V \oplus V^*)$

Let $V = \mathbb{C}[z, z^{-1}]$ and $V^* = \mathbb{C}[z, z^{-1}]$ be the restricted dual of $V$. Recall that

$$C\ell(V \oplus V^*) = \mathbb{C}\langle \psi_i, \psi_i^* \mid i \in \mathbb{Z}\rangle / \left\{\psi_i\psi_j + \psi_j\psi_i = 0, \quad \psi_i^*\psi_j^* + \psi_j^*\psi_i^* = 0, \quad \psi_i\psi_j^* + \psi_j^*\psi_i = \delta_{i+j=0}\right\}.$$

Further, recall that $\psi_i \leftrightarrow z^i$ and $\psi_i^* \leftrightarrow z^{i-1}\, \mathrm{d}z$.
Now we define the fermion space to be the semiinfinite wedge product.

**Definition 30** (fermion space).

We define the fermion space $\mathcal{F}$ to be

$$\begin{aligned} \mathcal{F} = \Lambda^{\frac{\infty}{2}+\bullet}(\mathbb{C}[z, z^{-1}]) = C\ell/\mathcal{I}, \text{ where} \\ \mathcal{I} = \text{left ideal generated by } \psi_i \text{ for } i \geq 0 \text{ and } \psi_i^* \text{ for } i > 0. \end{aligned}$$

This space has a monomial basis which can be indexed as follows:

$$\left\{\prod_{i \in S_1} \psi_i \prod_{j \in S_2} \psi_j^* \mid S_1 \subset \mathbb{Z}_{<0}, S_2 \subset \mathbb{Z}_{\leq 0}, \text{both finite}\right\}.$$

Therefore we have an alternative description as:

$$\mathcal{F} = \operatorname{Span}\{v_{i_1} \wedge v_{i_2} \wedge \cdots \mid i_1 < i_2 < \dots,\ \{i_1, i_2, \dots\} \setminus \mathbb{Z}_{\geq 0},\ \mathbb{Z}_{\geq 0} \setminus \{i_1, i_2, \dots\} \text{ are finite}\}.$$

Now we may identify $1 \leftrightarrow \psi_0 \wedge \psi_1 \wedge \psi_2 \wedge \dots$, and $\psi_i \leftrightarrow \psi_i \wedge -$ and $\psi_i^* \leftrightarrow \partial_{\psi_{-i}}$. Under this correspondence, we have that

$$\prod_{i \in S_1} \psi_i \prod_{j \in S_2} \psi_j^* \leftrightarrow \pm \bigwedge_{k \in (\mathbb{Z}_{\geq 0} \setminus -S_2) \cup S_1}.$$

## Gradings on $\mathcal{F}$

### Charge

**Definition 31** (charge).

We define **charge** to be

$$\deg \bigwedge_{k\in S} \psi_k := |S \setminus \mathbb{Z}_{\geq 0}| - |\mathbb{Z}_{\geq 0} \setminus S|.$$

Here, $\deg \psi_0 \wedge \psi_1 \wedge \cdots = 0$, $\deg \psi_i = 1$, and $\deg \psi_i^* = -1$ (as operators).

According to this, we have the grading on $\mathcal{F}$ as follows:

$$\mathcal{F} = \lambda^{\frac{\infty}{2}+\bullet}(V) = \bigoplus_k \underbrace{\Lambda^{\frac{\infty}{2}+k}}_{\text{charge}=k}.$$

### Energy

**Definition 32** (energy).

We define **energy** to be

$$\deg \bigwedge_{k\in S} \psi_k = -\sum_{k\in S\setminus \mathbb{Z}_{\geq 0}} k + \sum_{k\in \mathbb{Z}_{\geq 0}\setminus S} k.$$

Here, $\deg \psi_0 \wedge \psi_1 \wedge \cdots = 0$, and $\deg \psi_i = -i = \deg \psi_i^*$ (as operators).

**Remark 33.**

Charge can be any integer, but energy is always nonnonnegative (on the infinite wedges)!

## The mapping $\mathfrak{gl}_\infty \to C\ell(U)$

The space $\mathfrak{gl}_\infty = \mathfrak{gl}(\mathbb{C}[z, z^{-1}])$ has a basis of $E_{ij}$ (where the $i$th basis element of $\mathbb{C}[z, z^{-1}]$ is $z^i$). Then $\mathfrak{gl}_\infty \ni E_{ij} \mapsto \psi_i \psi_{-j}^* \in C\ell(U)$.

**Lemma 34.**

By an easy computation, we can see that

$$[\psi_i\psi^*_{-j}, \psi_k] = \begin{cases} 0 & k \neq j, \\ \\ \psi_i & k = j, \end{cases}$$

$$[\psi_i\psi^*_{-j}, \psi^*_{-k}] = \begin{cases} 0 & k \neq i, \\ \\ \psi^*_{-j} & k = i. \end{cases}$$

All this gives us the standard formula:

$$[E_{ij}, E_{kl}] = E_{il}\delta_{j=k} - E_{kj}\delta_{l=i}.$$

# Sep 20

## $\Lambda^{\frac{\infty}{2}+\cdot}(V)$, central extensions of $\mathfrak{gl}_\infty$

**Definition 35.**

Let $\mathfrak{gl}_\infty^f$ consist of endomorphisms of $V = \mathbb{C}[z, z^{-1}]$ with finitely many nonzero coefficients in the basis $\{v_i = z^i \mid i \in \mathbb{Z}\}$.
Let $\mathfrak{gl}_\infty^J \supset \mathfrak{gl}_\infty^f$ consist of "generalized Jacobi matrices" - endomorphisms of $V$ with nonzero coefficients on finitely many diagonals, as illustrated below (imagine that it goes off to infinity in every direction):

$$\begin{pmatrix}
* & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & * & * & * & * & * & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & * & * & * & * & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & * & * & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & * & * & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & *
\end{pmatrix}$$

**Definition 36** (Witt algebra)**.**

Let $V = \mathbb{C}[z, z^{-1}] = \text{span}\{z^k\}$ and $W = \text{span}\{L_i := -z^{i+1}\frac{\partial}{\partial z}\}$ be the **Witt algebra**. In the Witt algebra we have the relations $[L_n, L_m] = (n-m)L_{n+m}$.

We have

$$\mathfrak{gl}_\infty^f \subset \mathfrak{gl}_\infty^J \supset V, W.$$

The $\mathbb{Z}$-grading on $\mathfrak{gl}_\infty^f$, $\mathfrak{gl}_\infty^J$ comes from that on $V$, where $\deg z^i = i$. Therefore $\deg E_{ij} = i - j$, and this extends to $\mathfrak{gl}_\infty^J$ (which restricts to $V, W$).

Now suppose that $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$, and $\mathfrak{g}$ contains the subalgebras $\mathfrak{g}_{<0}$, $\mathfrak{g}_0$, and $\mathfrak{g}_{>0}$ (it is the direct sum of them). Note that this choice is not unique, nor is it canonical. However, we must choose one to define category $\mathcal{O}$.
As an example, for a finite-dimensional semisimple $\mathfrak{g}$, grading comes from $\mathrm{ad}_h$, where $h$ is part of a principal $\mathfrak{sl}_2$ triple. Then $\mathfrak{g}_{<0} = \mathfrak{n}_-$, $\mathfrak{g}_{>0} = \mathfrak{n}_+$, and $\mathfrak{g}_0 = \mathfrak{h}$. Then $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$ is just the Cartan decomposition.
Then we can consider the category $\mathcal{O}$ of $\mathfrak{g}$-modules.

**Definition 37.**

Let $\mathfrak{g}$ be a graded Lie algebra. A $\mathfrak{g}$-module $M$ is in **category-$\mathcal{O}$** if:

- $M$ is graded, upper bounded degree, and has finite dimensional graded components;
- $M = \bigoplus_{i\in\mathbb{Z}} M_i$ where $M_i = 0$ for $i \gg 0$, $\dim M_i < \infty$, and $\mathfrak{g}_i M_j \subset M_{i+j}$;
- $\mathfrak{g}_{>0}$ acts locally nilpotently (implied by the first point; conversely, along with some other mild assumption i.e. finitely generated, implies the first point);
- $\mathfrak{g}_0$ acts semisimply (this is a standard requirement for category $\mathcal{O}$, but is not implied by any of the previous points).

Just as in the finite-dimensional case, $\mathcal{U}(\mathfrak{g})$ acts on any $M \in \mathcal{O}$.

**Example 38.**

Let $M(\lambda)$ be the Verma module for $\mathfrak{sl}_2$. Consider the sum

$$1 + fe + f^2e^2 + f^3e^3 + \ldots \notin \mathcal{U}(\mathfrak{sl}_2).$$

However, this is a well-defined action on $M(\lambda)$, because for any given element of $M(\lambda)$, only finitely many terms in this infinite sum act nontrivially. This can be formalized as follows.
Let $J_{>N}$ be the left ideal in $\mathcal{U}(\mathfrak{g})$ generated by $\mathfrak{g}_{>N} := \bigoplus_{i>N} \mathfrak{g}_i$. We have that $J_{>N} \supset J_{>N+1}$, and $\bigcap_{N\in\mathbb{Z}} J_{>N} = 0$. Then consider

$$\widetilde{\mathcal{U}(\mathfrak{g})} := \lim_{\infty \leftarrow N} \mathcal{U}(\mathfrak{g})/J_{>N}.$$

Another option to get a bigger completion, is to define $J_{>N}$ to be the left ideal generated by

$\mathcal{U}(\mathfrak{g})_{>N}$. The difference between this definition and $\widetilde{\mathcal{U}(\mathfrak{g})}$ is that in $\widetilde{\mathcal{U}(\mathfrak{g})}$, the infinite series must be of bounded degree; in this alternate bigger completion, the degree can be unbounded.

Similarly, we can define a completion $\widetilde{C\ell}$ if $C\ell = C\ell(\mathbb{C}[z, z^{-1}] \oplus \mathbb{C}[z, z^{-1}\, dz])$, where $\deg \psi_i, \psi_i^* = i$.

**Proposition 39.**

We have an embedding of Lie algebras $\mathfrak{gl}_\infty^f \hookrightarrow C\ell$. This is done by $E_{ij} \mapsto \psi_i \psi_{-j}^*$. We also have an embedding $\widetilde{\mathfrak{gl}}_\infty^J \hookrightarrow \widetilde{C\ell}$, where $\widetilde{\mathfrak{gl}_\infty^J}$ is a central extension of $\mathfrak{gl}_\infty^J$.

The first embedding fails to extend to an embedding of $\mathfrak{gl}_\infty^J$. Imagine we have some infinite sum $\sum_{i+j=k} b_{ij} \psi_i \psi_j^*$ for some fixed $k$. We want finitely many terms outside the ideal $C\ell(\psi_{>N}, \psi_{>N}^*)$. We want to swap the factors $\psi_i$ and $\psi_j^*$ so that we have positive indices on the right. If $k \neq 0$, then there is no issue, since $\psi_i \psi_j^* = -\psi_j^* \psi_i$. The problem arises when $k = 0$: we have to sum up infinitely many constants. This is the reason why the central extension arises.

The solution is that we must change the original embedding $\mathfrak{gl}_\infty^J \hookrightarrow C\ell$: we send

$$E_{ij} \mapsto \begin{cases} \psi_i \psi_{-j}^* & j < 0 \\ -\psi_{-j}^* \psi_i & j \geq 0. \end{cases}$$

This is not an embedding of Lie algebras, but it is a homomorphism from a central extension of Lie algebras. (If we change the cutoff from $0$ to something else, say $1$, we achieve more or less the same result.) This is an embedding of a **trivial** central extension of $\mathfrak{gl}_\infty^f$. However, it extends to an embedding $\widetilde{\mathfrak{gl}}_\infty^J \hookrightarrow \widetilde{C\ell}$.
To compute the cocycle, we wish to compute the new commutators of the embedded elements

$$[E_{ij}, E_{kl}]_{\text{new}} = \delta_{j=k} E_{il} - \delta_{i=l} E_{kj} + \omega(E_{ij}, E_{kl}),$$

where the cocycle $\omega$ is nonzero iff $k = j$ and $l = i$. Then it turns out that

$$\omega(E_{ij}, E_{kl}) = \delta_{i=l} \delta_{k=j} \cdot \begin{cases} 0 & i, j \geq 0, \\ 0 & i, j < 0, \\ 1 & i \geq 0, j < 0, \\ -1 & i < 0, j \geq 0. \end{cases}$$

This is well-defined on generalized Jacobi matrices: on a generalized Jacobi matrix as follows,

$$
\begin{pmatrix}
\ast & \ast & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\ast & \ast & \ast & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \ast & \ast & \ast & \ast & \ast & {\color{red}\ast} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & {\color{red}\ast} & {\color{red}\ast} & {\color{red}\ast} & {\color{red}\ast} & \ast & \ast & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & {\color{red}\ast} & {\color{red}\ast} & {\color{red}\ast} & \ast & \ast & \ast & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & {\color{red}\ast} & {\color{red}\ast} & \ast & \ast & \ast & \ast & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & {\color{red}\ast} & \ast & \ast & \ast & \ast & \ast & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast
\end{pmatrix}
$$

there may be infinitely many nonzero entries, but the only entries which produce nonzero cocycles come from the region in red (bounded by the axes), which is finite.

**Remark 40.**

This cocycle is nicknamed the "Japanese cocycle" because it was first introduced by Date, Jimbo, Kashiwara, and Miwa sometime in the 1980s.

In the second homework, we compute the restrictions of this cocycle to the abelian Lie algebra and to the Witt algebra.

**Example 41.**

Let's compute the central extension of the abelian algebra $V = \mathbb{C}[z, z^{-1}]$.
Let $a_i = z^i$. Then $a_i$ consists of a single diagonal of 1s shifted by $i$. For example, $a_4$ looks like:

$$
\begin{pmatrix}
 & & 1 & & & & & & & & \\
 & & & 1 & & & & & & & \\
 & & & & 1 & & & & & & \\
 & & & & & 1 & & & & & \\
 & & & & & & 1 & & & & \\
 & & & & & & & 1 & & & \\
 & & & & & & & & 1 & & \\
 & & & & & & & & & 1 & \\
 & & & & & & & & & & 1 \\
 & & & & & & & & & & \\
 & & & & & & & & & &
\end{pmatrix}
$$

Then $[a_i, a_j]_{\text{new}} = \underbrace{[a_i, a_j]_{\text{old}}}_{=0} + i\delta_{i+j=0} \cdot 1$.

Then we get a central extension called the **Heisenberg algebra**:

$$0 \to \mathbb{C} \to \underbrace{\mathfrak{a}}_{\text{Heisenberg algebra}} \to V \to 0.$$

**Remark 42.**

An important observation is that $\Lambda^{\frac{\infty}{2}+\bullet}(V)$ is a category $\mathcal{O}$ object in $C\ell$-mod. This implies that

$$\mathfrak{a}, \mathrm{Vir} \subset \widetilde{\mathfrak{gl}}_{\infty}^{J} \curvearrowright \Lambda^{\frac{\infty}{2}+\bullet}(V),$$

where the Virasoro algebra Vir is the unique (up to isomorphism) nontrivial central extension of the Witt algebra $W$.

# Sep 25

## Heisenberg algebra action on $\Lambda^{\frac{\infty}{2}} V$

Recall that we defined the fermion space $\mathcal{F}$ in Definition 30 (fermion space).

**Remark 43.**

Unsurprisingly, $\mathcal{F}$ is related to fermions.

Now $\widetilde{C\ell} \curvearrowright \mathcal{F}$, with $\psi_i = v_i \wedge -$, and $\psi^*_{-i} = \partial_{v_i}$.

Recall from Example 41 that we computed the central extension of the abelian Lie algebra $\mathbb{C}[z, z^{-1}]$.

**Definition 44** (Heisenberg algebra)**.**

We define the Heisenberg algebra $\mathfrak{a}$ to be the central extension of the abelian Lie algebra

$$0 \to \mathbb{C}c \to \mathfrak{a} \to \mathbb{C}[z, z^{-1}] \to 0.$$

Explicitly,

$$\overline{\mathfrak{gl}}_{\infty}^{J} \supset \mathfrak{a} = \mathrm{Span}\{c, a_i \mid i \in \mathbb{Z}\}, \quad [c, -] = 0, \quad [a_i, a_j] = \delta_{i+j=0} c.$$

**Definition 45** (Fermion space as a Heisenberg algebra module).

We defined the Fermion space $\mathcal{F}$ as a semiinfinite wedge product, and a quotient of the Clifford algebra, in ^b98ac3. We now define the $\mathfrak{a}$-module structure on it.

The action of $a_i$ on $\mathcal{F}$ is given by

$$a_i \mapsto \sum_{r+s=i} \psi_r \psi_s^* \underbrace{=}_{\text{if } i \neq 0} \sum_{s>0, r+s=i} \psi_r \psi_s^* - \sum_{s \leq 0, r+s=i} \psi_s^* \psi_r,$$

with this alternate form if $i \neq 0$. When defined in this way, $a_0$ can act by any scalar, and we may choose this scalar. However, there is a natural choice for $a_0$ following this definition: we send

$$a_0 \mapsto \sum_{s>0} \psi_{-s} \psi_s^* - \sum_{s \leq 0} \psi_s^* \psi_{-s}.$$

## Is $\mathfrak{a} \curvearrowright \mathcal{F}$ irreducible?

Since $\mathfrak{a}$ acts on $\mathcal{F}$, the first question we ask is: is this representation irreducible?
**NO.**
We have two gradings: charge and energy, see Definition 31 (charge) and Definition 32 (energy).
Charge is by eigenvalues of $a_0$, so

$$\deg v_{i_1} \wedge v_{i_2} \wedge \cdots = |I \setminus \mathbb{Z}_{\geq 0}| - |\mathbb{Z}_{\geq 0} \setminus I|.$$

The charge comes from comparison to the vacuum vector $v_0 \wedge v_1 \wedge \ldots$.
This grading splits $\mathcal{F} = \bigoplus_{m \in \mathbb{Z}} \mathcal{F}_m$, and this grading is preserved by $\mathfrak{a}$, because everything in the Heisenberg algebra commutes with $a_0$. In particular, each $\mathcal{F}_m$ is a subrepresentation.

The next question: is $\mathcal{F}_m$ irreducible?
Now let us first take a brief detour to discuss what representations of the Heisenberg algebra $\mathfrak{a}$ look like in general.

# Category $\mathcal{O}$ for $\mathfrak{a}$

We first have the grading on $\mathfrak{a}$ as follows: $\deg a_i = i$, and $\deg c = 0$.

Then we write

$$\mathfrak{a} = \underbrace{\mathfrak{n}_-}_{\mathrm{Span}\{a_i | i<0\}} \oplus \underbrace{\mathfrak{h}}_{\mathrm{Span}\{a_0, c\}} \oplus \underbrace{\mathfrak{n}_+}_{\mathrm{Span}\{a_i | i>0\}}.$$

**Definition 46** (category-$\mathcal{O}(\mathfrak{a})$).

The objects of category-$\mathcal{O}$ for the Heisenberg algebra $\mathfrak{a}$ are $\mathfrak{a}$-modules $M$ which are:

- graded, $M = \bigoplus M_i$, $M_i$ are finite-dimensional and $a_i M_j \subset M_{i+j}$,
- bounded, i.e. $M_i = 0$ for $i \gg 0$,
- $\mathfrak{h}$-semisimple.

**Definition 47** (Fock space)**.**

The model objects in category-$\mathcal{O}(\mathfrak{a})$ are the "Verma" modules induced from one-dimensional $\mathfrak{h} \oplus \mathfrak{n}_+$ representations $\mathbb{C}_\chi$

$$F_\chi := \mathcal{U}(\mathfrak{a}) \otimes_{\mathcal{U}(\mathfrak{h}\oplus\mathfrak{n}_+)} \mathbb{C}_\chi,$$

where $\mathbb{C}_\chi$ satisfies the following properties:

- $\dim \mathbb{C}_\chi = 1$,
- $\mathfrak{n}_+ \mathbb{C}_\chi = 0$.
  We call the $F_\chi$ **Fock spaces**.

Now $\chi : \mathfrak{h} \to \mathbb{C}$ is determined by two numbers: $A$, which is the eigenvalue of $\mathfrak{a}_0$, and $C$, which is the eigenvalue of $c$. Let us denote $\chi$ by $(A, C)$.

As graded vector spaces, $F_\chi = \mathbb{C}[a_{-1}, a_{-2}, \ldots]$, the "symmetric polynomials of infinitely many variables." Making this identification, what is the action of $\mathfrak{a}$ on this space? The answer is that

$$a_i \xrightarrow{\text{acts by}} \begin{cases} a_i \cdot - & i < 0, \\ A \cdot - & i = 0 \\ C \cdot i \cdot \partial_{a_{-i}} & i > 0. \end{cases}$$

This is a very explicit and concrete description of this module. From this description, it is easy to see that:

**Proposition 48.**

If $C \neq 0$, then $F_\chi$ is irreducible.

***Proof.***
Suppose we have some polynomial $p \in \mathbb{C}[a_{-1}, a_{-2}, \ldots]$, where $p = a_{-1}^{u_1} a_{-2}^{u_2} \cdots a_{-N}^{u_N}$ is (one of) the

highest degree monomials.
Then

$$\partial_{a_{-1}}^{u_1} \cdots \partial_{a_{-N}}^{u_N} P \in \mathbb{C} - \{0\}, \quad \mathcal{U}(\mathfrak{a})1 = F_\chi.$$

## Block decomposition

□

It follows that $\mathcal{O} = \bigoplus_\chi \mathcal{O}_\chi$.

**Proposition 49.**

Let $\chi = (A, C)$. Then $C \neq 0 \implies \mathcal{O}_\chi$ contains unique simple object $F_\chi$.

***Proof.***
$M \ni v$ contains a highest degree vector $v$ satisfying:

- $\mathfrak{n}_+ v = 0$
- $\mathfrak{a}_\circ v = Av$
- $\mathfrak{c}\mathfrak{v} = Cv$.
  Then $F_\chi \to M$ sends $1 \mapsto v$.
  This has to be an embedding as $\mathcal{F}_\chi$ is simple, hence any $M \in \mathcal{O}_\chi$ contains a copy of $\mathcal{F}_\chi$.
  Furthermore, the $A, C$ are uniquely determined by $\chi$, hence the result.

□

**Proposition 50.**

In fact, $\mathcal{O}_\chi$ is a semisimple category and every object is a direct sum of $\mathcal{F}_\chi$s, i.e.
$\mathrm{Ext}^1_{\mathcal{U}(\mathfrak{a})}(F_\chi, F_\chi) = 0$.

***Proof.***
Suppose we had an extension

$$0 \to F_\chi \to M \to F_\chi \to 0.$$

Then take any homogeneous preimage $v'$ of $1 \in \mathcal{F}_\chi$ in $M$. On the other hand, $\mathcal{F}_\chi \ni 1 \mapsto v \in M$, and $v, v'$ are linearly independent highest weight vectors. Each of them generates a copy of $\mathcal{F}_\chi$ in $M$, and they do not intersect (as otherwise they would intersect in a proper submodule, contradicting the fact that $\mathcal{F}_\chi$ is simple).

□

**Remark 51.**

This is a standard argument when working in category $\mathcal{O}$.

Now crucially, this implies that all $\mathcal{F}_m$, and even $\mathcal{F}$, are category $\mathcal{O}$-representations. It remains to show that there are finitely many elements of a fixed energy.

We know that energy is nonnegative, and moreover there is a unique vector with minimal energy $0$: namely the vacuum vector $\psi := v_0 \wedge v_1 \wedge \cdots$. Now we can get any vector by applying $\psi_i$ for $i < 0$, which has degree $-i > 0$, and $\psi_i^*$ for $i \leq 0$, which still has (nonnegative) degree $-i \geq 0$. This implies that there are only finitely many monomial vectors with a given positive degree. This satisfies the condition that the graded components $\mathcal{F}_\chi$ are finite-dimensional.

But the graded components $\mathcal{F}_m$ are graded by charge, not energy! Fortunately, we see that $\mathcal{F}$ is a direct sum of $\mathcal{F}_\chi$, where $\chi = (A, 1)$ for different $A$'s (note that $c$ always acts by $1$, hence $C = 1$, but $\mathfrak{a}_0$ gives the charge degree). In particular, $\mathcal{F}_m = F_{m,1}^{\oplus N}$, so the $\mathcal{F}_m$ are also finite dimensional, thus $\mathcal{F}_m$ and $\mathcal{F}$ are category $\mathcal{O}$ objects.

Next time: we compare the characters to find the $N$ (spoiler: they are actually always $1$). We will compute

$$\mathrm{ch}\, \mathcal{F} = \sum t^{\text{charge}} q^N \dim \mathcal{F}_m[N],$$

where $m$ is the charge and $N$ is energy, and same for $F_{m,1}$.
This will give us some nontrivial combinatorial formula, which is known as the Jacobi triple product identity.

# Sep 27

## Some combinatorics

Recall Definition 30 (fermion space)

$$\mathcal{F} := \Lambda^{\frac{\infty}{2}+\bullet}(V) = C\ell(\psi_i, \psi_i^*)_{i\in\mathbb{Z}} / C\ell \cdot \{\psi_i, \psi_j^* \mid i \geq 0, j > 0\} \ni 1 = \Psi = v_0 \wedge v_1 \wedge \ldots.$$

(Here the $\mathcal{F}$ stands for “fermion.”)

## Gradings:

Recall also the charge and energy gradings: Definition 31 (charge) and Definition 32 (energy).

### Charge

The charge grading satisfies $C(\Psi) = 0$, $C(\psi_i) = 1$, and $C(\psi_i^*) = -1$.

### Energy

The energy grading satisfies $e(\Psi) = 0$, $e(\psi_i) = -i = e(\psi_i^*)$.
The energy degree on $\mathcal{F}$ is always nonnegative! This is because $\mathcal{F} = \Lambda(\psi_i, \psi_j^* \mid i < 0, j \leq 0\} \cdot \Psi$
.

Therefore

$$\mathcal{F} = \bigoplus_{m \in \mathbb{Z}, k \in \mathbb{Z}_{\geq 0}} \mathcal{F}_{m,k},$$

where $m$ is the charge and $k$ is the energy.

**Definition 52** (character of $\mathcal{F}$).

We can write the **character** of $\mathcal{F}$, a generating function, as

$$\mathrm{ch}\mathcal{F} = \sum \dim \mathcal{F}_{m,k} t^m q^k \in \mathbb{Z}[t, t^{-1}][[q]].$$

**Proposition 53.**

$$\mathrm{ch}\mathcal{F} = \left( \prod_{i>0} (1 + tq^i) \right) \left( \prod_{j \geq 0} (1 + t^{-1} q^j) \right).$$

## $\mathcal{F}$ as an $\mathfrak{a}$-mod

Recall from ^07753e that we have that

$$\mathcal{F} = \bigoplus_{\chi} F_{\chi}$$

as an $\mathfrak{a}$-mod, where $\mathfrak{a}$ is the Heisenberg algebra. The space $F_\chi$ is called the **Fock space** representation of the Heisenberg algebra $\mathfrak{a}$.

Recall that $\chi$ is determined by a pair $(A, C)$, where $A$ is the $a_0$-eigenvalue and $C$ is the $c$-eigenvalue. In $\mathcal{F}$, we have that $C = 1$ always, and $A = m$ when $\mathcal{F}_\chi \subset \mathcal{F}_m$, the charge. So in fact

$$\mathcal{F} = \bigoplus_{m \in \mathbb{Z}} F_{m,1}^{\oplus N_m}.$$

Our first task is to determine $\mathrm{ch} F_{m,1}$.

Note that as a vector space,

$$F_{m,1} = \mathbb{C}[a_{-1}, a_{-2}, \ldots] \cdot 1,$$

where $1$ has charge $m$ (and some unspecified energy).
Let us recall that

$$a_r = \sum_{i+j=r} \psi_i \psi_j^*,$$

hence the charge of each $a_r$ is $0$ (it is a sum of monomials with exactly one $\psi_i$ and one $\psi_j^*$, thus has $1-1$ charge), while the energy of each $a_r$ is $-r$; it follows that $a_{-1}$ has energy $1$, $a_{-2}$ has energy $2$, etc.
So $1$ has charge $m$ and unspecified energy $e$, while the character of $\mathbb{C}[a_{-1}, a_{-2}, \ldots]$ which is a symmetric algebra with generators of degree $1, 2, 3, \ldots$ hence has character $\prod_{i>0} \frac{1}{1-q^i}$.
Therefore we have that

$$\mathrm{ch} F_{m,1} = \sum_{e \in ?} ? \cdot t^m q^e \cdot \prod_{i>0} \frac{1}{1-q^i},$$

and we still need to determine the energy $e$.

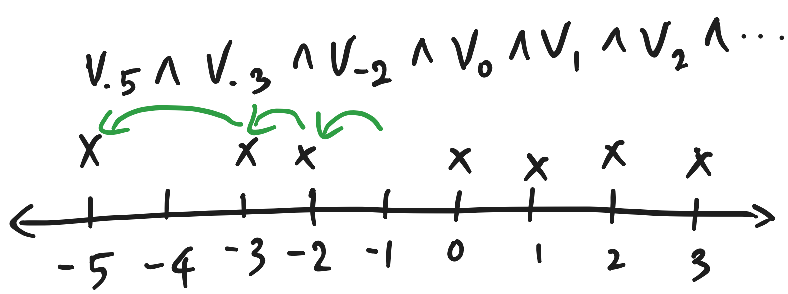
Therefore we want to find the relevant coefficients $N_m$ such that

$$\left(\prod_{i>0}(1+tq^i)\right)\left(\prod_{j\geq 0}(1+t^{-1}q^j)\right) = \sum_{m\in\mathbb{Z}, e=?} ? \cdot t^m q^e \prod_{i>0} \frac{1}{1-q^i}.$$

(We still don't know what's going on with $e$.)

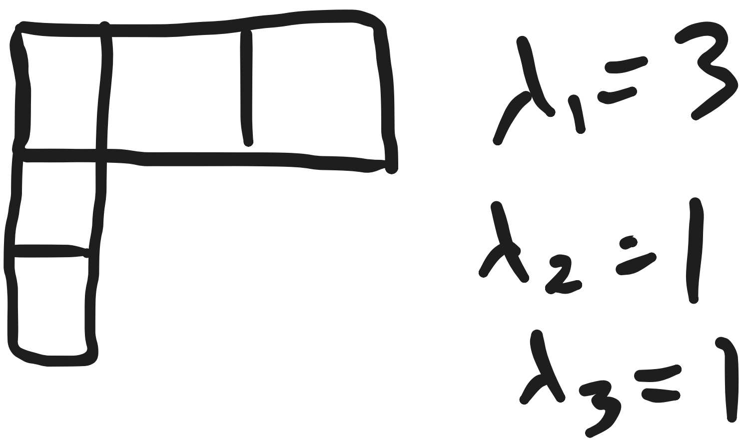
## Describing all monomials in $\mathcal{F}_{m,k}$

Fix $m \in \mathbb{Z}$. Now we describe all monomials in $\mathcal{F}_{m,k}$.
**Observation**: let us describe the minimal $k$ such that $\mathcal{F}_{m,k} \neq 0$.
Suppose $m \geq 0$. Then the minimal energy of a vector is the "dense" monomial $v_{-m} \wedge v_{-m+1} \wedge \cdots \wedge v_{-2} \wedge v_{-1} \wedge v_0 \wedge v_1 \wedge \ldots$, with no holes. The energy of this is $k = \frac{m(m+1)}{2}$.
**Observation**: Every possible monomial (of a fixed charge) can be produced from a "dense" monomial (with no holes) by moving some of the indices to a lower index. (Depicted below)




How does the energy change upon shifting these indices? We may assign $\lambda_1$ to be the distance the first vector moves, $\lambda_2$ the distance the second vector moves, etc. We observe that

$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots$ and for all sufficiently large $i$, $\lambda_i = 0$. (In the above example, $\lambda_1 = 2$, $\lambda_2 = \lambda_3 = 1$, and all remaining $\lambda_i = 0$.)
This means that to any monomial, we may assign a partition - equivalently, a Young diagram! In the above example, we have:




We know from the first observation that the minimal energy of any monomial with charge $m$ is $\frac{m(m+1)}{2}$. How much larger is the energy of a monomial with associated Young diagram $\mathcal{T}$? Well, then answer is simply $|\mathcal{T}|$: the number of boxes. In other words, the energy of a monomial with charge $m$ is $\frac{m(m+1)}{2} + |\mathcal{T}|$. (In the above example, $m = 3$ and $|\mathcal{T}| = 5$, so the energy is $11$.) It follows that partitions of $d$ give us the charge-$m$-monomials with energy $d$ more than the minimal energy. This is summarized below.

**Proposition 54.**

Let $P(d)$ denote the number of partitions of $d$. Then

$$\mathrm{ch}\mathcal{F}_m = t^m q^{\frac{m(m+1)}{2}} \sum P(d) q^d = t^m q^{\frac{m(m+1)}{2}} \prod_{i>0} \frac{1}{1-q^i}.$$

But now we are done! Because then $\mathcal{F}_m$ has the same size as the Fock space $F_{m,1}$, so $\mathcal{F}_m \cong F_{m,1}$ and all multiplicities are $1$.

## Jacobi triple product identity

**Theorem 55** (Jacobi triple product identity)**.**

$$\prod_{i>0}(1+tq^i) \prod_{j\geq 0}(1+t^{-1}q^j) \prod_{k>0}(1-q^k) = \sum_{m\in\mathbb{Z}} t^m q^{\frac{m(m+1)}{2}}.$$

**Corollary 56.**

$\mathcal{F}_m$ is irreducible as an $\mathfrak{a}$-module.

## Some consequences of Jacobi triple product identity

The Jacobi triple product identity is

$$\prod_{i>0}(1+tq^i)\prod_{j\geq 0}(1+t^{-1}q^j)\prod_{k>0}(1-q^k)=\sum_{m\in\mathbb{Z}}t^m q^{\frac{m(m+1)}{2}}.$$

If we expand $\prod_{k>0}(1-q^k)$, we will see that most of the coefficients are zero, and the remaining coefficients are $\pm 1$, precisely at the pentagonal numbers, due to Euler.
The precise formula, known as the pentagonal number theorem, is:

**Theorem 57** (Euler pentagonal number theorem)**.**

$$\prod_{k>0}(1-x^k)=\sum_{m\in\mathbb{Z}}(-1)^m x^{\frac{m(3m+1)}{2}}.$$

It turns out that this can be deduced from the Jacobi triple product identity by substituting $q=x^3$ and $t=-x^{-1}$. This is a very important identity!

Now consider the negative part of the Witt algebra,

$$W_+ := \operatorname{Span}\{L_1, L_2, \ldots\}\subset W.$$

This is a maximal nilpotent subalgebra of $W$, and the gradings are given by $\deg L_i = i$ (from the action of $\operatorname{ad} L_0$). Now recall from Sep 13 that the cohomology of a maximal nilpotent subalgebra is related to BGG resolutions.
Then $H^\bullet(W_+)=H(\Lambda^\bullet(W_+^*))$, where $\Lambda^\bullet(W_+^*)$ has one generator in degree $1$, one generator in degree $2$, etc. So the complex $\Lambda^\bullet(W_+^*)$ is graded both by the $\bullet$ and by the generators from the space itself. The character of this complex is precisely $\prod_{k\geq 0}(1-x^k)$.

**Theorem 58** (L. Goncharova)**.**

For $m>0$,

$$\dim H^m(W)=2,$$

and the degrees are precisely the values $\frac{m(3m\pm 1)}{2}$.

**Remark 59.**

The proof was later simplified by Feigen-Fuks using the Laplace operator.

More precisely, the complex $\Lambda^\bullet(W_+^*)$ obtains a grading from $\operatorname{ad} L_0$ and the cohomology groups inherit this grading. It turns out that $\dim H^m(W) = 2$ for any $m > 0$, and the nonzero graded components are precisely the values $\frac{m(3m\pm1)}{2}$.

# Sep 29

Today we will discuss the combinatorics from last class, and explain the relation of the theorem from last class to symmetric polynomials.

## Euler pentagonal number formula

Recall from last class, Theorem 57 (Euler pentagonal number theorem) tells us that

$$\prod_{i>0}(1-x^i) = \sum_{m\in\mathbb{Z}}(-1)^m x^{\frac{m(3m+1)}{2}}.$$

Last time we proved this identity by substituting particular values into the Jacobi triple product identity. Let us see a combinatorial proof of this identity.

***Proof.***

The idea is that the left hand side is some Euler characteristic.

First,

$$\prod_{i>0}(1-x^i) = \sum_{n=0}^{\infty}(q_+(n) - q_-(n))x^n,$$

where $q_+(n)$ denotes the number of partitions of $n$ into an even number of different parts, and $q_-(n)$ denotes the number of partitions of $n$ into an odd number of different parts.

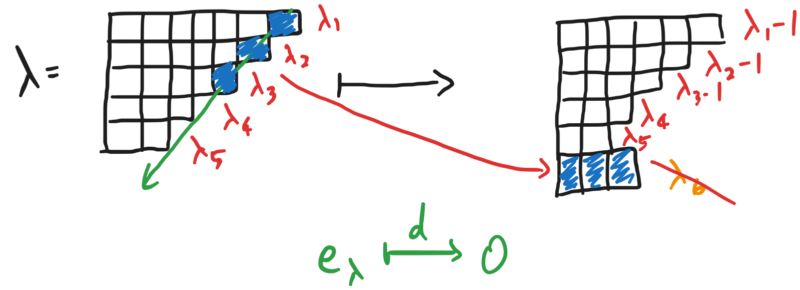
Let $q(n,k)$ be the number of partitions of $n$ into $k$ different parts, i.e.

$q(n,k) := \#\{n = \lambda_1 + \cdots + \lambda_k \mid \lambda_1 > \lambda_2 > \cdots > \lambda_k > 0\}$. It follows that $q_+(n) = \sum q(n, 2m)$ and $q_-(n) = \sum q(n, 2m+1)$.

Consider the graded vector space

$$C^\bullet(n) = \bigoplus C^k(n), \quad C^k(n) = \mathbb{C}^{q(n,k)}, \quad \text{basis given by partitions as above.}$$

Let us define the differential $d : C^\bullet(n) \to C^{\bullet+1}(n)$. Let $e_\lambda$ denote the basis element associated to the partition $\lambda$ of $n$. Define $\lambda'$ to be the partition constructed as follows. Consider a Young tableaux associated to $\lambda$ (as below). We then take boxes off of the furthest diagonal, as far as we can, and then move them to the bottom to form a new row. If we cannot do this (i.e., if the new bottom row is not shorter than the original bottom row and thus is not a valid Young tableaux with strictly decreasing rows), then set $d(e_\lambda) = 0$. Otherwise, set $d(e_\lambda) = e_{\lambda'}$.




It turns out that $d^2 = 0$. This is because the diagonal of the newly formed Young tableaux is at least the length of the diagonal of the original Young tableaux.
Now what is $H^\bullet(n)$, the cohomology of this complex? It is clear that $H^k(n)$ is the number of $\lambda$ such that $d(\lambda) = 0$ and there does not exist $\widetilde{\lambda}$ such that $d(\widetilde{\lambda}) = \lambda$. The first condition, $d(\lambda) = 0$, is equivalent to the length of the diagonal being at least the length of the bottom row. The second condition, $\lambda \notin d(C^{k-1})$, is equivalent to the condition that we cannot put the last row as the diagonal. But in fact there are only two possibilities illustrated below: it is an $m \times m$ square with a staircase attached in two possible ways:


These give $m^2 + \frac{m(m-1)}{2} = \frac{m(2m-1)}{2}$ and $m^2 + \frac{m(m+1)}{2} = \frac{m(3m+1)}{2}$, respectively.
Now the left hand side of Euler pentagonal number identity is just the generating function of the Euler characteristic of this complex, while the right hand side is the generating function for the cohomology of the complex.

□

**Remark 60.**

What is unknown about this (and is a very interesting question) is whether we can reduce the computation of $H(W_+)$, the Witt algebra, to some complex using this method.

## $\Lambda^{\frac{\infty}{2}+\bullet}(V) = \mathcal{F} = \bigoplus_{m\in\mathbb{Z}} F_m$

Recall (see Definition 30 (fermion space), Definition 47 (Fock space)) that $\mathcal{F}$ is the fermion space, $F_m$ is the Fock space (aka boson space), and the $F_m$ are the decomposition according to the charge grading. (This is known as the boson-fermion correspondence.)

This is a bit surprising: something which is an exterior algebra looks like something which is a direct sum of symmetric algebras (recall that $F_m$ is a free module over a symmetric algebra, namely $F_m = \mathbb{C}[a_{-1}, a_{-2}, \ldots] \cdot 1$). This is impossible in finite dimensional case, as the exterior part is finite dimensional while the symmetric part is infinite dimensional. But in the infinite-dimensional case, this is actually not that surprising!

**Example 61** (easiest infinite-dimensional example).

Consider $\mathbb{C}[x]$. Then $\mathrm{Sym}^n(\mathbb{C}[x]) = (\mathbb{C}[x]^{\otimes n})^{S_n} = \mathbb{C}[x_1, \ldots, x_n]^{S_n}$ which is the algebra of symmetric polynomials. But recall that this is just $\mathbb{C}[e_1, e_2, \ldots, e_n]$ where $e_k$ is the $k$th elementary symmetric polynomial (of degree $k$); this is also equal to $\mathbb{C}[p_1, \ldots, p_n]$ where $p_k = \sum_{j=1}^n x_j^k$ is the $k$th power sum; there are many other presentations but they all have generators in degree $1, 2, \ldots, n$ .
Now $\Lambda^n(\mathbb{C}[x]) = \mathbb{C}[x_1, \ldots, x_n]^{\text{skew-symmetric}}$ (i.e. $p(x_1, \ldots, x_n)|_{x_i = x_j} = 0$). But then each polynomial is divisible by $\prod_{i>j}(x_i - x_j)$, and furthermore the quotient is a symmetric polynomial, hence

$$\Lambda^n(\mathbb{C}[x]) = \prod_{i>j}(x_i - x_j)\mathbb{C}[x_1, \ldots, x_n]^{S_n},$$

so the two spaces differ by just a grading, induced by multiplication by the factor $\prod_{i>j}(x_i - x_j)$.
Now we need the basis in $\mathbb{C}[x_1, \ldots, x_n]^{S_n}$ given by the monomial basis in $\Lambda^n(\mathbb{C}[x])$: these are precisely alternations of $x_1^{\lambda_1+n-1} x_2^{\lambda_2+n-2} \cdots x_n^{\lambda_n}$ for all possible partitions $\lambda := (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$. But this is the same as the determinant of the matrix given by $(a_{ij} = x_j^{\lambda_i+n-i})$. Now the factor from above is just the determinant of the Vandermonde determinant of the matrix $(b_{ij} = x_j^{i-1})$, so the basis elements in $\mathbb{C}[x_1, \ldots, x_n]^{S_n}$ corresponding to the monomial basis in $\Lambda^n(\mathbb{C}[x])$ is just the ratio of these two determinants. We call these basis elements $s_\lambda(x_1, \ldots, x_n)$ the **Schur polynomials**. These form a natural basis of $\mathbb{C}[x_1, \ldots, x_n]^{S_n}$!
The Schur polynomials can be regarded as a polynomial in $\{e_1, \ldots, e_n\}$, or in $\{p_1, \ldots, p_n\}$, or in any other generating set.

**Proposition 62.**

The presentation of the Schur polynomial does not depend on $n$, i.e.

$$s_\lambda(x_1, \ldots, x_n) = S_\lambda(p_1, p_2, \ldots).$$

Now recall that the charge $0$ part of the fermion space $\mathcal{F}$, namely $\mathcal{F}_0$, is identified with the Fock space $F_0 = \mathbb{C}[a_{-1}, \ldots]$. It has a monomial basis, $\Psi_\lambda$ for all possible partitions $\lambda$, where

$\Psi_\lambda = v_{-\lambda_1} \wedge v_{-\lambda_2+1} \wedge \cdots \wedge v_{-\lambda_n+n-1} \wedge v_n \wedge \ldots$, where $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)$.
We'd like to identify $\Psi_\lambda$ as a polynomial in the $a_{-j}$, using the identification $F_0 = \mathbb{C}[a_{-1}, a_{-2}, \ldots]$.

**Theorem 63.**

$\Psi_\lambda = S_\lambda(a_{-1}, a_{-2}, \ldots, a_{-n}, \ldots)\Psi_0.$

The proof will be next time, but the proof uses the presentation of Schur polynomials above, namely the interpretation of Schur polynomials as a monomial basis in $\Lambda^n(\mathbb{C}[x])$.

# Oct 2

## Boson-fermion correspondence $\longleftrightarrow$ symmetric polynomials

Let's first recall some properties of the ring of symmetric polynomials. Recall that

$$\mathbb{C}[x_1, \ldots, x_N]^{S_N} = \mathbb{C}[e_1, \ldots, e_n], \quad e_m = \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq N} x_{i_1} x_{i_2} \cdots x_{i_m}.$$

The polynomial $e_m$ is called the $m$th elementary symmetric polynomial.
However, this is not the only presentation. We have other generators:

$$\begin{aligned} h_m &= \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq N} x_{i_1} \cdots x_{i_m}, \quad \text{complete symmetric polynomial,} \\ p_m &= \sum_{i=1}^N x_i^m, \quad \text{power sums.} \end{aligned}$$

**Proposition 64.**

$\mathbb{C}[x_1, \ldots, x_N]^{S_N} = \mathbb{C}[h_1, \ldots, h_N] = \mathbb{C}[p_1, \ldots, p_N].$

**Remark 65.**

Note that the first equality is true over $\mathbb{Z}$, but the second is only true over $\mathbb{Q}$! So power sums require more than just integers to achieve equality.

***Proof.***
We want to express the $e_i, h_i$ in terms of $p_i$. To do this, we want to write the generator series for each set of generators.
We have the generating series

$$\prod_{i=1}^{N}(1-x_i t) = 1 + \sum (-1)^m e_m t^m,$$

$$\prod_{i=1}^{N}(1-x_i t)^{-1} = 1 + \sum h_m t^m.$$

So we have that

$$(1 - e_1 t + e_2 t^2 - \ldots)(1 + h_1 t + h_2 t^2 + \ldots) = 1.$$

From this we may write the $h_i$ in terms of $e_i$; for example, $h_1 = e_1$, $h_2 = e_1^2 - e_2$, etc.
Now we may write the generating series for $p_m$ as

$$\sum p_m t^{m-1} = \sum_{i=1}^{N} \sum_{m=1}^{\infty} x_i^m t^{m-1} = \sum_{i=1}^{N} \frac{x_i}{1 - x_i t}.$$

To achieve something of this form, we apply $\partial_t \log$ to the generating series for $e_m$.

$$\frac{\sum (-1)^m m e_m t^{m-1}}{1 + \sum (-1)^m e_m t^m} = \partial_t \log \prod (1 - x_i t) = \sum_{i=1}^{N} \frac{x_i}{1 - x_i t} = \sum_m p_m t^{m-1}.$$

It follows that

$$\sum (-1)^m m e_m t^{m-1} = \left(\sum p_m t^{m-1}\right)\left(1 + \sum (-1)^m e_m t^m\right).$$

Now using this identity, we can express $e$ in terms of $p$, and vice versa.

□

We have surjections

$$\pi_N : \mathbb{C}[x_1, \ldots, x_N, x_{N+1}]^{S_{N+1}} \xrightarrow{x_{N+1}=0} \mathbb{C}[x_1, \ldots, x_N]^{S_N}$$

which are homomorphisms of graded algebras.
Furthermore, for $m < N+1$, we have $e_m \mapsto e_m$, and the kernel of this map is $(e_{N+1})$.
This means that we can define the inverse limit of the symmetric algebras.

**Definition 66** (infinite symmetric algebra)**.**

Define

$$\mathbb{S} := \text{graded} \lim_{\infty \leftarrow N} \mathbb{C}[x_1, x_2, \ldots, x_N]^{S_N} = \mathbb{C}[e_1, e_2, \ldots, e_m, \ldots].$$

(We require the inverse limit over the graded components so as not to obtain infinite series.)

**Proposition 67.**

$\pi_N(h_m) = h_m$ and $\pi_N(p_m) = p_m$, so we have well-define elements $p_m, h_m \in \mathbb{S}$, hence

$$\mathbb{S} = \mathbb{C}[h_1, h_2, \ldots] = \mathbb{C}[p_1, p_2, \ldots].$$

## Natural bases of $\mathbb{S}$

We have $\mathbb{C}[x_1, \ldots, x_N]^{S_N} \ni$ monomial symmetric functions $m_\lambda$ for partitions $\lambda = \lambda_1 \geq \lambda_2 \geq \ldots, \geq \lambda_N$ of length $N$. By definition,

$$m_\lambda := \sum x_{i_1}^{\lambda_1} \cdots x_{i_N}^{\lambda_N},$$

where the sum runs over all possible monomials of the above form.
**Warning!!** this is NOT just the symmetrization; for example if two $\lambda_i$ are equal, then we only count that monomial once, e.g. for $N = 2$ and $(1,1)$ partition, we have $m_{1,1} = e_2 = x_1 x_2$, NOT $x_1 x_2 + x_2 x_1$.

Now $\mathbb{C}[x_1, \ldots, x_N]^{S_N}$ contains the Schur polynomials $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$. This comes from identifying $\mathbb{C}[x_1, \ldots, x_N]^{S_N} \cong \mathbb{C}[x_1, \ldots, x_N]^{S_N} \prod_{i>j}(x_i - x_j)$ of skew-symmetric functions, and the latter has a monomial skew-symmetric basis identified with Schur polynomials.

**Proposition 68.**

Recall from Sep 29 that $s_\lambda$ is a ratio of two determinants. Then the leading term is $m_\lambda$ and the remaining terms are smaller:

$$s_\lambda = m_\lambda + \sum_{\mu < \lambda} (*) \cdot m_\mu.$$

In other words, the $s_\lambda$ is upper triangular in the basis of $m_\mu$.

**Proposition 69.**

$\pi_N(m_\lambda) = m_\lambda$ and $\pi_N(s_\lambda) = s_\lambda$. This implies that we have bases $m_\lambda, s_\lambda \in \mathbb{S}$ indexed by **all** partitions (in contrast, the discussion above limits the partitions to $N$).

**Theorem 70.**

We have an isomorphism

$$\mathbb{S} \cong F_0 = \mathbb{C}[a_{-1}, a_{-2}, \ldots], \quad \mathbb{S} \ni p_i \mapsto a_{-m}.$$

This is clear, as they are both just graded rings of polynomials in infinitely many variables. But furthermore, under the isomorphism $F_0 = \mathcal{F}_0$, the monomial basis $\Psi_\lambda$ of $\mathcal{F}_0$ corresponds to the Schur polynomial basis $s_\lambda$ of $F_0$.

***Proof.***
Consider the following finite approximation.

$$\Lambda^N(\mathbb{C}[z^{-1}] \cdot z^{N-1}) \hookrightarrow \mathcal{F}_0,$$

sending

$$\xi \mapsto \xi \wedge v_N \wedge v_{N+1} \wedge \ldots.$$

This is an isomorphism on all graded components of this space of energy $\leq N$. Furthermore, on these graded components, the morphism commutes with the action of
$\mathfrak{a}_- = \mathrm{Span}\{a_{-1}, a_{-2}, \ldots\} \subset \mathfrak{a}$.
Moreover, identifying $z^{-1} \leftrightarrow x$, we have

$$\Lambda^N(\mathbb{C}[z^{-1}]z^{N-1}) \cong \mathbb{C}[x_1, \ldots, x_N]^{S_N} \cdot \prod_{i>j}(x_i^{-1} - x_j^{-1}),$$

where monomials correspond to monomial skew-symmetric functions.
Finally, we note that $a_{-m}$ acts as multiplication by $p_m$ on $\mathbb{C}[x_1, \ldots, x_N]^{S_N} \prod_{i>j}(x_i^{-1} - x_j^{-1})$.
Why is this? Well, $\left(\mathbb{C}[z^{-1}]z^{N-1}\right)^{\otimes N} = \mathbb{C}[x_1, \ldots, x_N] \cdot x_1^{1-N} \cdots x_N^{1-N}$, and the elements of this Lie algebra act by the Leibniz rule:

$$z^{-m} \mapsto z^{-m} \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes z^{-m} \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes z^{-m} = (x_1^m + \cdots + x_N^m) = p_m.$$

On the other hand, the monomial basis is in terms of the Schur polynomial basis of $\mathbb{C}[x_1, \ldots, x_N]^{S_N} \prod_{i>j}(x_i^{-1} - x_j^{-1})$. This proves the theorem for energy level up to $N$. But this works for any $N$, so we are done.

□

# Oct 4

## Boson-fermion correspondence, continued

Recall that

$$\mathcal{F} = \Lambda^{\frac{\infty}{2}+\bullet} V = \bigoplus_{m \in \mathbb{Z}} \mathcal{F}_m = \bigoplus_{m \in \mathbb{Z}} F_m,$$

where $F_m$ are the (irreducible) Fock space representations of $\mathfrak{a}$ and $\mathcal{F}$ is an irreducible representation of $C\ell$.

We have already expressed the action of $a_i \in \mathfrak{a}$ by elements in the Clifford algebra, via $a_i \mapsto \sum_{r+s=i} \psi_r \psi_s^*$, written in normal ordering, by $\psi_r \psi_s^*$ for $s > 0$ and $-\psi_s^* \psi_r$ for $s \leq 0$. Our preliminary question is whether we can express the $\psi_i$ and $\psi_i^*$ in terms of the $a_i$. This is certainly not possible, as the $a_i$ preserve the $F_m$ while the $\psi_i$ do not! However, we have other natural operators, namely $z, z^{-1}$ which are "shift" operators shifting the highest weight vectors $\mathcal{F}_m \ni \Psi_m \mapsto \Psi_{m+1} \in \mathcal{F}_{m+1}$. Our objective now is thus to define an action of $z : \Psi_m \mapsto \Psi_{m+1}$, commuting with all $a_i$ for $i \neq 0$. Indeed, excluding $a_0$, all of the $F_m$ are isomorphic as $\mathfrak{a}$-modules, and $z, z^{-1}$ are the operators inducing this isomorphism.

**Question:** Express $\psi_i$, $\psi_i^*$ in terms of $z$ and $a_i$ for $i \in \mathbb{Z}$.

**Remark 71.**

Since $\Psi_m = v_{-m} \wedge v_{-m+1} \wedge \ldots$, the action of $z$ is completely determined and well-defined.

Let's collect some facts we know about these operators.

1. $[a_i, \psi_i] = \psi_{i+j}$.
2. $[a_i, \psi_j^*] = -\psi_{i+j}^*$.

**Remark 72.**
This can be seen by the action of $a_i \leftrightarrow z^i$ as an abelian Lie algebra.

**Lemma 73.**

The maps $\psi_j : F_m \to F_{m+1}$ are uniquely determined by:

1. $\Psi_{m+1} = \psi_{-m-1} \Psi_m$ and $\psi_r \Psi_m = 0$ for $r \geq -m$.
2. $[a_i, \psi_j] = \psi_{i+j}$.

***Proof.***
Let's first give an example of how this works. The operator $\psi_{-m-1}$ is already defined on $\Psi_m$, the highest weight vector. First let us note that

$$\begin{aligned}\psi_{-m}(a_{-1}\Psi_m) &= a_{-1}\psi_{-m}\Psi_m - [a_{-1}, \psi_{-m}]\Psi_m, \\ &= 0 - \psi_{-m-1}\Psi_m, \\ &= -\Psi_{m+1}.\end{aligned}$$

Similarly, we can determine $\psi_j$ on any monomial of strictly negative terms (i.e. all $i_j > 0$) $a_{-i_1} \cdots a_{-i_r} \Psi_m$, for $j - i_1 - \cdots - i_r = -m - 1$.

Now if we want to determine say, $\psi_{-m-2}\Psi_m$, we see that it will be something proportional to $a_{-1}\Psi_{m+1}$, and we can find this coefficient by applying $a_1$ to both sides: we have

$$\begin{aligned} \psi_{-m-2}\Psi_m &= ka_{-1}\Psi_{m+1}, \\ a_1\psi_{-m-2}\Psi_m &= k\Psi_{m+1}, \\ \psi_{-m-1}\Psi_m &= k\Psi_{m+1} \implies k = 1. \end{aligned}$$

Although this proves that the maps are uniquely determined, it doesn't give any explicit formulas! Now we want to get explicit formulas. □

**Definition 74.**

Let us write the formal generating series

$$\psi(u) := \sum_{j\in\mathbb{Z}} \psi_j u^{-j}, \quad \psi^*(u) := \sum_{j\in\mathbb{Z}} \psi_j^* u^{-j}.$$

These satisfy commutator relations

$$[a_i, \psi(u)] = u^i\psi(u), \quad [a_i, \psi^*(u)] = -u^i\psi^*(u).$$

**Idea**: We want to write $\psi(u) = \Gamma(u, a_i)_{i\in\mathbb{Z}}$. Suppose all the $a_i$ are commutative and we replace $a_i$ with $i\frac{\partial}{\partial a_{-i}}$ (the motivation is that $[a_i, a_j] = i\delta_{i+j=0}$, which essentially acts by $i\frac{\partial}{\partial a_{-i}}$). Then $[a_{-i}, -] = \frac{\partial}{\partial a_i}$. So

$$\frac{\partial}{\partial a_i}\Gamma(u, a_i) = \frac{u^{-i}}{i}\Gamma(u, a_i),$$

which has a candidate solution

$$\Gamma(u, a_i) = f(u)\exp(\sum_{i\neq 0}\frac{a_i}{i}u^{-i}),$$

which is great if the $a_i$ commute, but not well defined if the $a_i$ do not commute.

Instead, let's consider

$$\exp\left(\sum_{i<0}\frac{a_i}{i}u^{-i}\right)\cdot\exp\left(\sum_{i>0}\frac{a_i}{i}u^{-i}\right),$$

which is a well-defined operator on $F_m((u))$ for any Fock space $F_m$. (This is indeed Laurent series because only finitely many terms in the right sum act nonzero.)

So from the previous "highly incorrect" considerations, we constructed a well-defined operator!

**Theorem 75.**

$$\Gamma(u) = u^{m+1} z \exp\left(\sum_{i>0} -\frac{a_{-i}}{i} u^i\right) \cdot \exp\left(\sum_{i>0} \frac{a_i}{i} u^{-i}\right) = \psi(u),$$

$$\Gamma^*(u) = u^{-m} z^{-1} \exp\left(\sum_{i>0} \frac{a_{-i}}{i} u^i\right) \cdot \exp\left(\sum_{i>0} -\frac{a_i}{i} u^{-i}\right) = \psi^*(u).$$

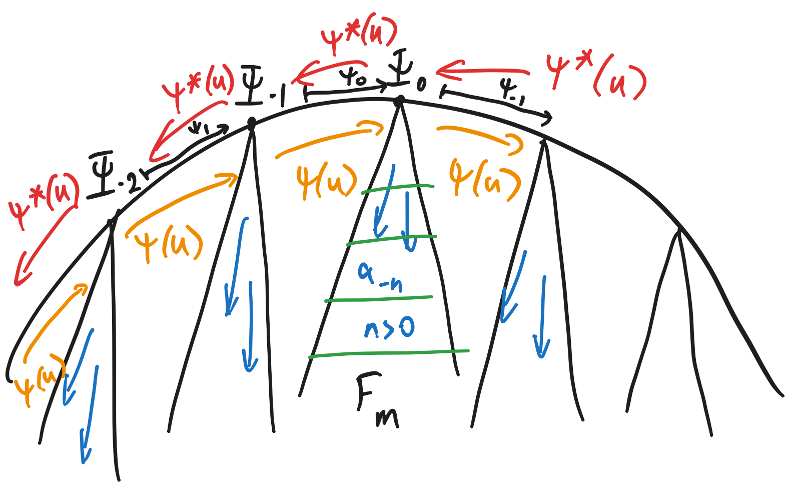
***Proof.***
The proof is fairly easy; we only need to check the relations $[a_i, \psi(u)] = u^i \psi(u)$ and $[a_i, \psi^*(u)] = -u^i \psi^*(u)$, and that $\Gamma(u)\Psi_m = u^{m+1}\Psi_{m+1} + \text{smaller order terms}$. Let's do the commutator relations first. We need to check three separate cases: $i = 0$, $i < 0$, and $i > 0$.
For $i = 0$, the two exponentials commute, but the $z^{\pm 1}$ doesn't commute. But using that $[a_0, z] = z$, we find that $[a_0, \Gamma(u)] = \Gamma(u)$ (and similarly for $\Gamma^*(u)$).
For $i < 0$, the $u^{m+1}z$ term commutes, the first exponential commutes, but the last exponential doesn't; however, we already checked that $[a_i, -]$ acts by $\frac{\partial}{\partial_{a_{-i}}}$.
The same thing occurs for $i > 0$.
For the action on $\Psi_m$, clearly $u^{m+1}z + \Psi_m = u^{m+1}\Psi_{m+1}$, while the second exponential acts by identity and the first exponential only contains positive powers of $u$, hence $\Gamma(u)\Psi_m = u^{m+1}\Psi_{m+1} + u^{m+2}\mathbb{C}[[u]]F_{m+1}$. Now by the lemma and explicitly checking the relations, we can verify the above expressions for $\psi$ and $\psi^*$.

□

# Oct 6

## Boson-fermion correspondence and Jacobi-Trudy(-Giambelli) identities

Recall that we have $\mathcal{F} = \bigoplus_{m\in\mathbb{Z}} F_m$, decomposition into Fock spaces. It's helpful to keep the following picture in mind:



Each Fock space $F_m$ is generated by the highest vector $\Psi_m$, via the $a_{-n}$ for $n > 0$. The $a_{>0}$ move the vectors upwards. This module is cyclic over $\mathfrak{a}_-$, and cocyclic with respect to $\mathfrak{a}_+$ (everything can be taken to $\Psi_m$ by $a_{>0}$ elements). The fermion space is direct sum of such things, and is graded with respect to energy. Now the height of an element in this picture is just its degree with respect to energy, and the vertices belong to some parabola. Each Fock space is an $\mathfrak{a}$-module, and there is an action of $\psi(u)$ moving vectors clockwise, while $\psi(u)^*$ moves things counterclockwise.

Now we have the operator $\psi(u) = \sum \psi_n u^{-n}$ and is uniquely determined by the property that $\psi_{-m-1}\Psi_m = \Psi_{m+1}$ and $\psi_N \Psi_m = 0$ for $N > -m-1$, and the property that $[a_i, \psi(u)] = u^i \psi(u)$. These two properties allow you to uniquely determine the action of $\psi(u)$ between any two adjacent Fock spaces. First you determine all of the $\psi$ going to the highest vector, and then you use the properties to go down in the target Fock space. So these are uniquely determined.

**Theorem 76.**

We have the following formulas. The notation $u^{a_0}$ denotes the action by $u^m$ on $F_m$. The operator $z$ is the shift operator $F_m \to F_{m+1}$.

$$\psi(u) = u^{a_0} z \Gamma_-(u)\Gamma_+(u); \quad \Gamma_-(u) = \exp\left(\sum_{n>0} \frac{a_{-n}}{n} u^n\right),$$
$$\Gamma_+(u) = \exp\left(-\sum_{n>0} \frac{a_n}{n} u^{-n}\right).$$

$$\psi^*(u) = z^{-1} u^{-a_0} \Gamma_-^*(u)\Gamma_+^*(u); \quad \Gamma_-^*(u) = \exp\left(-\sum_{n>0} \frac{a_{-n}}{n} u^n\right),$$
$$\Gamma_+(u) = \exp\left(\sum_{n>0} \frac{a_n}{n} u^{-n}\right).$$

**Example 77.**

Consider $\psi(u) : F_{-1} \to F_0 \leftrightarrow \mathbb{S}$. Let us consider the action of $\psi(u) \cdot \Psi_{-1}$. Well first, $\Gamma_+(u)$ acts by identity because each of the $a_{>0}$ act trivially. Next, we have action of $\Gamma_-(u)$. Since $z$ commutes with all $a_{>0}$, we apply $z\Psi_{-1} = \Psi_0$ and $u^{a_0} = u^0 = 1$ on this space, so

$$\psi(u) : \Psi_{-1} \mapsto \exp\left(\sum_{n>0} \frac{a_{-n}}{n} u^n\right)\Psi_0.$$

On the other hand, we know that $\psi(u) = \sum \psi_{-n} u^n$, so the action is directly computed. Let $v := \Psi_{-1} = v_1 \wedge v_2 \wedge v_3 \wedge \ldots$. Then

$$\psi(u)\Psi_{-1} = v_0 \wedge v + u \cdot v_{-1} \wedge v + u^2 \cdot v_{-2} \wedge v + \dots.$$

Now recall that in the identification $F_0 \leftrightarrow \mathbb{S}$, $\Psi_0 \leftrightarrow 1$ and $a_{-n} \leftrightarrow p_n$, the power sum. Furthermore, the monomials correspond to Schur polynomials. In particular, the monomials above ( $v_{-n} \wedge v \leftrightarrow h_n$). So by comparing the actions of the exponential of the sum of $a_{-n}$ (converting $a_{-n} \leftrightarrow p_n$) with the directly computed action of $\psi(u)\Psi_{-1}$ via $v_{-n} \wedge - \leftrightarrow h_n$, the conclusion is that

$$\exp\left(\sum_{n>0} \frac{p_n}{n} u^n\right) = 1 + \sum_{n>0} h_n u^n = h(u) \in \mathbb{S}.$$

**Remark 78.**

In fact we've already seen this; taking the derivative of both sides, we get something that indeed we've before.

Now if we apply $\psi^*(u) : F_1 \to F_0$, then

$$\psi^*(u)\Psi_1 = \Gamma_-^*(u)\Psi_0.$$

On the other hand, $\psi^*(u) = \sum \psi_n^* u^{-n}$, and each $\psi_n^*$ works by deleting successive entries from $\Psi_1 = v_{-1} \wedge v_0 \wedge v_1 \wedge \dots$. So $\psi^*(u)\Psi_1$ is a sum of things corresponding to a one-column diagram (times some appropriate power of $u$, up to sign), hence

$$\Gamma_-^*(u) \leftrightarrow e(u) = 1 + \sum_{n>0} (-1)^n e_n u^n \in \mathbb{S}.$$

## Goal

Our goal is to use $\Gamma_+(u)$ to express any Schur polynomial $s_\lambda$ in the power sums $p_1, p_2, \dots$. Under the identification $\mathbb{S} \leftrightarrow F_0$, the Schur polynomial $s_\lambda$ for $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m \geq 0)$ corresponds to the monomial

$$\begin{aligned} s_\lambda \leftrightarrow & v_{-\lambda_1} \wedge v_{-\lambda_2+1} \wedge \dots \wedge v_{-\lambda_m+m-1} \wedge v_m \wedge \dots \\ &= \psi_{-\lambda_1}\psi_{-\lambda_2+1} \cdots \psi_{-\lambda_m+m-1}\Psi_{-m} \end{aligned}$$

where $\Psi_{-m} \in F_{-m}$. This is the coefficient of $u_1^{\lambda_1} u_2^{\lambda_2-1} \cdots u_m^{\lambda_m-m+1}$ in $\psi(u_1)\cdots\psi(u_m)\Psi_{-m}$. Now

$$\psi(u_1)\cdots\psi(u_m)\Psi_{-m} = u_1^{-0}u_2^{-1}\cdots u_m^{-m+1}\Gamma_-(u_1)\Gamma_+(u_1)\Gamma_-(u_2)\Gamma_+(u_2)\cdots\Gamma_-(u_m)\Gamma_+(u_m)\Psi_0 \in F_0.$$

However, these terms don't really commute… so we **want** all $\Gamma_+$ on the right.

**Observation.**

$$\left[\sum_{n>0} \frac{a_{-n}}{n} u^n, -\sum_{n>0} \frac{a_n}{n} u^{-n}\right] = \text{scalar operator}.$$

This is because this commutator is just an infinite sum of commutators $[a_{-n}, a_n]$ which is just a scalar.

Additionally, note that $[z, a_0] = z$.

**Lemma 79.**

Suppose $[A, B] = C$, $[C, A] = [C, B] = 0$. Then $e^A e^B = e^C e^B e^A$.

***Proof.***
First we note that $A^k B = BA^k + nCA^{n-1}$. Then we note that

$$A^n B^k = \sum_{l=0}^{\infty} l! \binom{n}{l} \binom{k}{l} C^l B^{k-l} A^{n-l}.$$

Now we have that

$$e^A e^B = -\sum \frac{A^n B^k}{n!k!} = \sum \frac{1}{n!k!} \frac{l! C^l B^{k-l} A^{n-l} k!}{l!^2 (n-l)!(k-l)!} = \sum_l \frac{C^l}{l!} \sum_k \frac{B^k}{k!} \sum_n \frac{A^n}{n!} = e^C e^B e^A.$$

□

**Corollary 80.**

Applying the previous lemma to the $\Gamma$, we find that

$$\Gamma_+(u_1)\Gamma_-(u_2) = \Gamma_-(u_2)\Gamma_+(u_1) \cdot \exp\left(-\sum_{n>0} \frac{1}{n}\left(\frac{u_2}{u_1}\right)\right) = \Gamma_-(u_2)\Gamma_+(u_1) \cdot \left(1 - \frac{u_2}{u_1}\right).$$

***Proof.***
Just note that the exponential is the logarithm power series, also that $[a_n, a_{-n}] = n$.

□

**Proposition 81.**

$s_\lambda$ is the coefficient of $u_1^{\lambda_1} \cdots u_m^{\lambda_m}$ in

$$\prod_{i=1}^m \exp\left(\sum_{n>0} \frac{p_n}{n} u_i^n\right) \prod_{i<j} \left(1 - \frac{u_j}{u_i}\right).$$

***Proof.***
We use the corollary to swap all of the $\Gamma_+$ to the right, which gives us the product of $\Gamma_-(u_i)$, but with the extra factor of $1-\frac{u_j}{u_i}$.

□

Now recall that $\exp\left(\sum_{n>0}\frac{p_n}{n}u_i^n\right)=h(u_i)$, while $\prod_{i<j}(1-\frac{u_j}{u_i}$ is the Vandermonde determinant (up to a monomial factor). Therefore the above expression equals $\prod_{i=1}^m h(u_i)\cdot u_1^{-m+1}u_2^{-m+2}\cdot\dots\cdot\det\left((a_{ij}=u_j^{m-i})_{i,j=1}^m\right)$. From this, we obtain:

**Theorem 82** (1st Jacobi-Trudy identity)**.**

$$s_\lambda=\det\left((h_{\lambda_i+j-i})_{i,j=1}^m\right)=\det\begin{pmatrix} h_{\lambda_1} & h_{\lambda_1+1} & h_{\lambda_1+2} & \dots & h_{\lambda_1+m-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & h_{\lambda_2+1} & \dots & h_{\lambda_2+m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_m-m+1} & h_{\lambda_m-m+2} & h_{\lambda_m-m+3} & \dots & h_{\lambda_m}\end{pmatrix}.$$

**Exercise 83.**

You can similarly express $s_\lambda$ as a determinant of the elementary symmetric functions (using the $e_i$ instead of the $h_i$) as well. The only difference is you transpose the Young diagram. The proof is the same, but you replace $\psi$ by $\psi^*$.

# Oct 11

## (untwisted) Affine Kac-Moody Lie algebras

Let us start with any simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, e.g. $\mathfrak{sl}_n$, $\mathfrak{so}_n$ for $n\geq 5$, $\mathfrak{sp}_{2n}$ for $n\geq 4$, exceptional ones, etc.

Fix an invariant inner product $\langle,\rangle$ on $\mathfrak{g}$ satisfying $\langle x,[y,z]\rangle=\langle[x,y],z\rangle$; since $\mathfrak{g}$ is simple, this inner product is unique up to scalar.

Now consider the **loop algebra** $\mathfrak{g}[z,z^{-1}]$. It's called loop because it's "equal" to $T_eLG$, where $LG=\{\text{analytic maps } S^1\to G\}$ (not precisely; we should really take some completion of $\mathfrak{g}[z,z^{-1}]$, but it's a dense subspace inside of $T_eLG$, which is enough for our purposes). This algebra $\mathfrak{g}[z,z^{-1}]$ is graded: the $n$th graded component is $\mathfrak{g}t^n$, hence the grading is given by operator $z\partial_z$. Hence $\mathfrak{g}[z,z^{-1}]=\bigoplus_{n\in\mathbb{Z}}\mathfrak{g}z^n$ is a graded Lie algebra, with $[\mathfrak{g}z^n,\mathfrak{g}z^m]=\mathfrak{g}z^{n+m}$.

As usual, we care about **central extensions** of this Lie algebra.

## Central extensions of $\mathfrak{g}[z,z^{-1}]$

Recall that central extensions are classified by $H^2(\mathfrak{g}[z, z^{-1}])$, so we want to compute this cohomology group. We have the Chevalley complex

$$C^\bullet(\mathfrak{g}[z, z^{-1}]) = \Lambda^\bullet(\mathfrak{g}[z, z^{-1}]^*)$$

and it carries an action of $\mathfrak{g}[z, z^{-1}]$ which acts trivially on cohomology. In particular, $\mathfrak{g}$ acts trivially on the cohomology.

**Proposition 84.**

Any cohomology class from $H^2(\mathfrak{g}[z, z^{-1}])$ is represented by a cocycle of the form $\sum_{m \neq n} \gamma_{m,n} \omega_{m,n}$ where $\gamma_{m,n} = -\gamma_{n,m} \in \mathbb{C}$ and $\omega_{m,n}(xz^k, yz^l) = \delta_{m=k}\delta_{n=l}\langle x, y\rangle$, i.e., a $\mathfrak{g}$-invariant cocycle.

***Proof.***
First, $\mathfrak{g} \curvearrowright H^2$ trivially iff we can lift any $\omega \in H^2$ to some $\mathfrak{g}$-invariant cocycle in $C^2$. Meanwhile, we have an embedding $\mathfrak{g}[z, z^{-1}]^* \hookrightarrow \Lambda^2 \mathfrak{g}[z, z^{-1}]^*$, so the coboundaries are precisely a copy of $\mathfrak{g}[z, z^{-1}]^*$. But $\mathfrak{g}[z, z^{-1}]^* = \bigoplus_{\mathbb{Z}} \mathfrak{g}$ contains no $\mathfrak{g}$-invariants, hence lifting is unique (there are no Ext between this module and the trivial module). It follows that $\mathfrak{g}$-invariant cocycles precisely have the above form.

□

This still consists of infinitely many parameters. We will whittle it down.

**Proposition 85.**

We have $\gamma_{n,m+p} + \gamma_{m,p+n} + \gamma_{p,n+m} = 0$.

***Proof.***
We have

$$\begin{aligned} 0 &= d\omega(x_1 z^n, x_2 z^m, x_3 z^p), \\ &= \omega([x_1, x_2] z^{n+m}, x_3 z^p) - \omega([x_1, x_3] z^{n+p}, x_2 z^m) + \omega([x_2, x_3] z^{m+p}, x_1 z^n), \\ &= \gamma_{n+m,p} \langle [x_1, x_2], x_3\rangle - \gamma_{n+p,m}\langle [x_1, x_3], x_2\rangle + \gamma_{m+p,n}\langle [x_2, x_3], x_1\rangle. \end{aligned}$$

Since $\langle , \rangle$s are totally antisymmetric, all of the $\langle , \rangle$ equal the same constant up to sign(note that $\langle [x_1, x_3], x_2\rangle = \langle x_1, [x_3, x_2]\rangle = -\langle x_1, [x_2, x_3]\rangle = -\langle [x_1, x_2], x_3\rangle$), hence the above expression is equal to some constant times $\gamma_{p,n+m} + \gamma_{m,p+n} + \gamma_{n,m+p}$, and choosing the constant to be nonzero we have that this is zero.

□

**Corollary 86.**

$\gamma_{n,-n} = n\gamma_{1,-1} = n\gamma$ and $\gamma_{m,n} = 0$ otherwise. In particular, $\dim H^2(\mathfrak{g}[z, z^{-1}]) = 1$.

***Proof.***

We have $\gamma_{n,s-n}+\gamma_{m,s-m}=\gamma_{n+m,s-n-m} \implies \gamma_{0,s}=0$ for all $s$. By induction, $-(s-n)\gamma_{1,s-1}=\gamma_{n,s-n}=n\cdot\gamma_{1,s-1} \implies (s-n)\gamma_{1,s-1}=-n\gamma_{1,s-1} \implies s\cdot\gamma_{1,s-1}=0$, hence $\gamma_{1,s-1}=0$ for all $s\neq 0$. This implies the result.

□

We finally reach our most important object in this course.

**Definition 87** (affine Kac-Moody Lie algebra)**.**

Any central extension of $\mathfrak{g}[z,z^{-1}]$ has the form

$$0\to\mathbb{C}c\to\widehat{\mathfrak{g}}\to\mathfrak{g}[z,z^{-1}]\to 0,\quad [xz^n,yz^m]_{\widehat{\mathfrak{g}}}=[x,y]_{\mathfrak{g}}z^{n+m}+n\cdot\delta_{n+m=0}\cdot\langle x,y\rangle\cdot c.$$

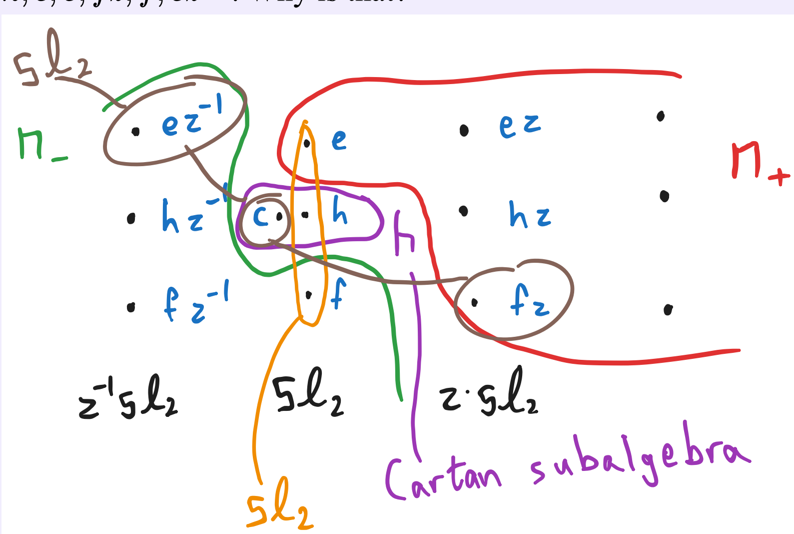
Equivalently,

$$[x(z),y(z)]_{\widehat{\mathfrak{g}}}=[x(z),y(z)]_{\mathfrak{g}}+\mathrm{Res}_{z=0}\langle x(z),\,\mathrm{d}y(z)\rangle\cdot c.$$

Therefore, we define the **(untwisted) affine Kac-Moody Lie algebra** $\widehat{\mathfrak{g}}$ associated to a simple (finite-dimensional) Lie algebra $\mathfrak{g}$ to be the unique (up to rescaling) nontrivial central extension of the loop space $\mathfrak{g}[z,z^{-1}]$.

**Example 88.**

Let $\mathfrak{g}=\mathfrak{sl}_2$. We may consider $\widehat{\mathfrak{sl}_2}$ as a bigraded Lie algebra, with one grading coming from $z\partial_z$, and the other grading coming from $\operatorname{ad}h$, as illustrated below. Then $\widehat{\mathfrak{sl}_2}$ is generated by $h,c,e,fz,f,ez^{-1}$. Why is that?




Anything to the right can be generated by $e$ and $fz$. For example, $hz=[e,fz]$ and $fz^2=[fz,hz]$. Anything to the left can be generated by $f$ and $ez^{-1}$ by a similar procedure.
Note that there are two copies of $\mathfrak{sl}_2$ here which indeed generate the whole thing, illustrated in

brown and orange. The relations are very similar to the relations coming from finite-dimensional Lie algebras.
This is very similar to the case of semisimple Lie algebras: we have the Cartan subalgebra in purple, $\mathfrak{n}_-$, $\mathfrak{n}_+$, illustrated in green and red, and we can define all the usual stuff such as roots, simple roots, Chevalley generators, etc., which we will discuss next time.

# Oct 13

## Structure and (some) representations of $\widehat{\mathfrak{sl}_2}$

Recall the construction from Definition 85 (affine Kac-Moody Lie algebra): to a complex semisimple Lie algebra $\mathfrak{g}$, we can assign a $\mathbb{Z}$-graded Lie algebra $\widehat{\mathfrak{g}}$ (called the affine Kac-Moody Lie algebra), which is a central extension of the Loop algebra $\mathfrak{g}[z, z^{-1}]$:

$$\widehat{\mathfrak{g}} = \mathbb{C}c \oplus \mathfrak{g}[z, z^{-1}], \quad [c, -] = 0, \quad [x(z), y(z)]_{\widehat{\mathfrak{g}}} = [x(z), y(z)]_{\mathfrak{g}[z,z^{-1}]} + \mathrm{Res}_{z=0}\langle x(z),\, \mathrm{d}y(z)\rangle c.$$

In fact, the inner product $\langle , \rangle$ is defined only up to a scalar; there are two ways to rectify this, either defining it to be the Killing form on $\mathfrak{g}$, or by normalizing $\langle h_i, h_i\rangle = 2$ (i.e., the square of any $h$ of a principal $\mathfrak{sl}_2$-triple is $2$).

### Question: What is $\mathrm{Der}\,\widehat{\mathfrak{g}}$?

**Remark 89.**

For a finite-dimensional simple Lie algebra, $\mathrm{Der}\,\mathfrak{g} = \mathrm{ad}\,\mathfrak{g}$ since $\mathrm{Der}\,\mathfrak{g}/\mathrm{ad}\,\mathfrak{g} = H^1(\mathfrak{g}, \mathfrak{g}) = 0$ (because the adjoint representation is a **nontrivial** irreducible representation, recall from Sep 11).

So we might want to study $\mathrm{Der}\,\widehat{\mathfrak{g}}/\mathrm{ad}\,\widetilde{\mathfrak{g}} = H^1(\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}})$. We have

$$\underbrace{\mathcal{D}\mathrm{er}\,\widehat{\mathfrak{g}}}_{\text{Lie algebra}} / \underbrace{\mathrm{ad}\,\widehat{\mathfrak{g}}}_{\text{ideal}} = H^1(\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}) \underbrace{\curvearrowleft}_{\text{trivially}} \mathfrak{g} \subset \widehat{\mathfrak{g}}.$$

So any class in $\mathcal{D}\mathrm{er}\,\widehat{\mathfrak{g}}/\mathrm{ad}\,\widehat{\mathfrak{g}}$ is represented by a $\mathfrak{g}$-invariant derivation. Moreover, $\mathcal{D}\mathrm{er}\,\widehat{\mathfrak{g}}$ is $\mathbb{Z}$-graded by $[z\partial_z, -]$, since $\widehat{\mathfrak{g}}$ is graded.

**Proposition 90.**

$\mathcal{D}\mathrm{er}\,\widehat{\mathfrak{g}}/\mathrm{ad}\,\widehat{\mathfrak{g}} = W = \mathbb{C}[z, z^{-1}]\partial_z$.

***Proof.***
Suppose $\mathcal{D} \in \mathcal{D}\mathrm{er}\,\widehat{\mathfrak{g}}$ is $\mathfrak{g}$-invariant, with $\deg \mathcal{D} = n$ homogeneous. Then for $x \in \mathfrak{g}$,

$$\mathcal{D}(xz^r) = \alpha_r x z^{r+n}, \quad \text{since } \mathcal{D} \text{ is } \mathfrak{g}-\text{invariant},$$

$$\begin{aligned} \mathcal{D}([xz^r, yz^s]) &= [\mathcal{D}(xz^r), yz^s] + [xz^r, \mathcal{D}(yz^s)], \\ \implies \alpha_{r+s}[x,y]z^{r+s+n} &= (\alpha_r + \alpha_s)[x,y]z^{r+s+n} \quad \forall\, x, y \in \mathfrak{g}, \\ \implies \alpha_{r+s} &= \alpha_r + \alpha_s. \end{aligned}$$

This means that any derivation in $\mathcal{D}\mathrm{er}\,\widehat{\mathfrak{g}}$ is uniquely determined up to a constant, and furthermore is proportional to $z^{n+1}\partial_z$.

□

**Definition 91** (extended affine Kac-Moody Lie algebra)**.**

It is sometimes useful to work with a somewhat bigger algebra, the **extended affine Kac-Moody algebra** $\widetilde{\mathfrak{g}}$, defined by

$$0 \to \widehat{\mathfrak{g}} \to \widetilde{\mathfrak{g}} \to \mathbb{C}\{z\partial_z\} \to 0.$$

We will often denote this extra element by $d := z\partial_z$.

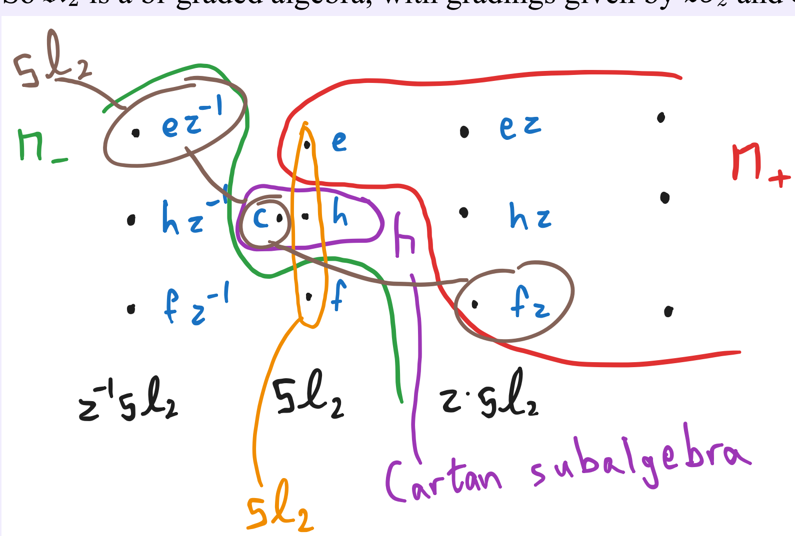
**Example 92.**

Let's examine $\widehat{\mathfrak{sl}_2}$. As we have already seen in Oct 11, we have the Cartan decomposition

$$\widehat{\mathfrak{sl}_2} = \widehat{\mathfrak{n}}_+ \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}_-,$$

$$\begin{aligned} \widehat{\mathfrak{h}} &= \mathrm{Span}\{h[0], c\}, \\ \widehat{\mathfrak{n}}_+ &= \mathbb{C}e[0] \oplus z\mathfrak{sl}_2[z], \\ \widehat{\mathfrak{n}}_- &= \mathbb{C}f[0] \oplus z^{-1}\mathfrak{sl}_2[z^{-1}]. \end{aligned}$$

So $\widehat{\mathfrak{sl}_2}$ is a bi-graded algebra, with gradings given by $z\partial_z$ and $\mathrm{ad}\, h$. Recall the picture from Oct 11:



Then we have a **principal grading** given by $2z\partial_z + \frac{1}{2}\operatorname{ad} h$, which gives a grading by the diagonals from top left to bottom right.

## Generators of $\widehat{\mathfrak{sl}_2}$

Consider $e_1 = e[0]$, $e_0 = f[1]$, $f_1 = f[0]$, $f_0 = e[-1]$, $h_1 = h[0]$, and $h_0 = c - h[0]$. Then $\{e_1, f_1, h_1\}$ and $\{e_0, f_0, h_0\}$ form two $\mathfrak{sl}_2$-triples. (You can easily check the commutator relations.)
Now note that $[e_1, f_0] = 0 = [e_0, f_1]$.
We also have analogues of Serre relations, easily seen by looking at the picture above (each commutator pushes it one step up or one step down, but there are only three levels):

$$[e_1, [e_1, [e_1, e_0]]] = 0,$$
$$[e_0, [e_0, [e_0, e_1]]] = 0.$$

We do have some for $f$s, but in fact, **we don't need other relations** and obtain $\widehat{\mathfrak{sl}_2}$ already through these few relations (though the proof of this will be postponed).

## Invariant symmetric bilinear form on $\widehat{\mathfrak{sl}_2}$

**Definition 93.**

Let us define the invariant symmetric bilinear form on $\widehat{\mathfrak{sl}_2}$:

- $\langle c, 0\rangle = 0$ (therefore the form is **degenerate**)
- $\langle e[r], f[-r]\rangle = 1$
- $\langle h[r], h[-r]\rangle = z$.

Question: What is the **Casimir element** here?
The problem is that this algebra is infinite-dimensional, so if you try to write the Casimir element, it will be an infinite sum, hence not an element of the universal enveloping algebra. Darn!
What we want to write is $\sum_{r\in\mathbb{Z}} e[r]f[-r] + f[r]e[-r] + \frac{1}{2}h[r]h[-r]$. But since we only consider category-$\mathcal{O}$ representations, $\widehat{\mathfrak{n}}_+$ acts locally nilpotently, so anything with positive terms on the right will act nilpotently.
While we **cannot make this element a finite sum**, we **can** make it well-defined on every category-$\mathcal{O}$ representation, and hence can view it as an element of the completion

$$\mathcal{O} \ni M \curvearrowleft \widetilde{\mathcal{U}(\widehat{\mathfrak{sl}_2})} = \lim_{\infty\leftarrow n} \mathcal{U}(\widehat{\mathfrak{sl}_2})\Big/ z^n\mathfrak{sl}_2[z].$$

So our goal is to commute the terms so the positive terms are on the right. We obtain

$$S=\left(e[0]f[0]+f[0]e[0]+\frac{1}{2}h[0]^2\right)+2\sum_{r>0}\left(e[-r]f[r]+e[-r]e[r]+\frac{1}{2}h[-r]h[r]\right).$$

The problem: $S$ is not central!
The good news: $[S,x[r]]=0$ modulo linear terms.

**Proposition 94.**

$[S,x[r]]=-2(c+2)r\cdot x[r]$.

The proof is deferred to next class, since we ran out of time!

From this, it will follow that $\mathcal{D}\mathrm{er}\,\widehat{\mathfrak{sl}_2}$ is generated by inner derivations and the Witt algebra $W=\mathbb{C}[z,z^{-1}]$. It will follow that we can recognize $z\partial_z=\left[\frac{S}{-2(c+2)},-\right]$ (after localizing), which will turn outer derivations to inner ones as well.

The true Casimir will be $S+2(c+2)\cdot z\partial_z\in\widetilde{\mathcal{U}(\widetilde{\mathfrak{sl}_2})}$, and this does indeed enjoy all of the properties of a true Casimir element.

# Oct 16

## Structure and representations of $\widehat{\mathfrak{sl}_2}$

Let us recall what we know about this affine Kac-Moody algebra. It is defined by

$$0\to\mathbb{C}c\to\widehat{\mathfrak{sl}_2}\to\mathfrak{sl}_2[z,z^{-1}]\to 0.$$

In particular there is a basis given by $c$ and $e[r],f[r],h[r]$ for $r\in\mathbb{Z}$. We have the relations

$$\begin{aligned}
[c,-]=0&=[e[r],e[s]]=[f[r],f[s]]\\
[h[r],h[s]]&=2r\delta_{r+s=0}c\\
[e[r],f[s]]&=h[r+s]\\
[h[r],e[s]]&=2e[r+s]\\
[h[r],f[s]]&=-2f[r+s]
\end{aligned}$$

We have the Cartan decomposition $\widehat{\mathfrak{sl}_2}=\widehat{\mathfrak{n}}_-\oplus\widehat{\mathfrak{h}}\oplus\widehat{\mathfrak{n}}_+$, where

$$\begin{aligned}
\widehat{\mathfrak{n}}_-&=\mathbb{C}f[0]\oplus z^{-1}\mathfrak{sl}_2[z^{-1}],\\
\widehat{\mathfrak{h}}&=\mathrm{Span}\{h[0],c\},\\
\widehat{\mathfrak{n}}_+&=\mathbb{C}e[0]\oplus z\mathfrak{sl}_2[z].
\end{aligned}$$

Let's distinguish some generators which play the role of Chevalley generators here. The natural choice is $e_1, e_0$ to generate $\widehat{\mathfrak{n}}_+$ and $f_1, f_0$ to generate $\widehat{\mathfrak{n}}_-$, along with $h_1, h_0$ to generate $\widehat{\mathfrak{h}}$. We define $e_1 = e[0]$, $f_1 = f[0]$, $h_1 = [e_1, f_1] = h[0]$, and $e_0 = f[1]$, $f_0 = e[-1]$, and $h_0 = [e_0, f_0] = c - h[0]$. Each triple $(e_i, f_i, h_i)$ generate an $\mathfrak{sl}_2$-triple.

In this way, we can regard $\widehat{\mathfrak{sl}_2}$ as a Lie algebra constructed from a Cartan matrix, just as in the finite-dimensional case. For $a_{ij} \in \mathbb{Z}$ with $i, j \in \{0, 1\}$, let

- $[h_i, e_j] = a_{ij} e_j$
- $[h_i, f_j] = -a_{ij} f_j$
- $[e_i, f_j] = \delta_{ij} h_i$
  So the Cartan matrix in our case is

$$A = (a_{ij}) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix},$$

  which is a degenerate Cartan matrix (this is in stark contrast to the finite-dimensional case, where the Cartan matrix is symmetrizable and the symmetrization is positive definite!). But we can still determine a Lie algebra from a Cartan matrix by imposing the above relations and the **Serre relations**:

$$(\operatorname{ad} e_i)^{1-a_{ij}} \cdot e_j = 0 = (\operatorname{ad} f_i)^{1-a_{ij}} \cdot f_j.$$

  As we have already seen, these relations already hold for $\widehat{\mathfrak{sl}_2}$! For $i \neq j$, then $1 - a_{ij} = 3$, and this is indeed the case. A bit later, we will see that $\widehat{\mathfrak{sl}_2}$ is indeed determined by these relations.

This sort of affine Kac-Moody Lie algebra generalizes. For any simple $\mathfrak{g}$, we can make the Lie algebra $\widehat{\mathfrak{g}} := \mathbb{C}c \oplus \mathfrak{g}[z, z^{-1}]$. We want to split the algebra

$$\cdots \oplus z^{-1}\mathfrak{g} \oplus (\mathfrak{g} \oplus \mathbb{C}c) \oplus z\mathfrak{g} \oplus z^2\mathfrak{g} \oplus \cdots$$

into a Cartan decomposition.

**Definition 95** (Cartan decomposition of affine Kac-Moody Lie algebra)**.**

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra. The Cartan decomposition of $\widehat{\mathfrak{g}}$ is as follows. We define

$$\begin{aligned} \widehat{\mathfrak{h}} &:= \mathfrak{h} \oplus \mathbb{C}c, \\ \widehat{\mathfrak{n}}_+ &:= \mathfrak{n}_+ \oplus z\mathfrak{g}[z], \\ \widehat{\mathfrak{n}}_- &:= \mathfrak{n}_- \oplus z^{-1}\mathfrak{g}[z^{-1}], \end{aligned}$$

where we view $\mathfrak{h}, \mathfrak{n}_+, \mathfrak{n}_- \subset \mathfrak{g} \cdot 1 \subset \mathfrak{g}[z, z^{-1}]$.

If we do this sort of Cartan decomposition, we want to determine the Chevalley generators, i.e. positive simple roots. We define the simple roots to be $e_i := e_{\alpha_i}[0] \in \mathfrak{n}_+ \subset \mathfrak{g}$. We can also define $e_0 = z \cdot f_\theta$ where $\theta$ is the maximal root; this is **one** extra root which lives in $z \cdot \mathfrak{g}$ and together with the simple roots, generates all of $\widehat{\mathfrak{n}}_+$. For example, in $\mathfrak{g} = \mathfrak{sl}_n$ we have $e_0 = z \cdot f_\theta = \begin{pmatrix} 0 & & \\ & \ddots & \\ z & & 0 \end{pmatrix}$, while $f_0 = \begin{pmatrix} 0 & & z^{-1} \\ & \ddots & \\ & & 0 \end{pmatrix}$. This is a Lie algebra which corresponds to the extended system of simple roots (i.e., all of the simple roots, combined with negative of the maximal root (alternatively, the lowest root)). You can still regard this as a system of simple roots! Then after that, we can write the Cartan matrix and the corresponding relations.

## Category $\mathcal{O}$ for $\widehat{\mathfrak{g}}$

Let $d : \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}$ be the derivation given by $z\partial_z$.

**Definition 96** (category-$\mathcal{O}$ for affine Kac-Moody Lie algebra)**.**

A $\widehat{\mathfrak{g}}$-module $M$ is in category $\mathcal{O}$ if:

1. it is $d$-graded (i.e. extends to $\widetilde{\mathfrak{g}}$-module).
2. $\widehat{\mathfrak{n}}_+$ acts locally nilpotently, i.e., for all $v \in M$, there exists $N$ such that $e_{i_1} \cdot \cdots \cdot e_{i_N} v = 0$.
3. $\widehat{\mathfrak{h}}$ acts semisimply, i.e. $M = \bigoplus_{\mu \in \widehat{\mathfrak{h}}^*} M_\mu$ such that for all $h \in \widehat{\mathfrak{h}}$ and $v \in M_\mu$, then $hv = \mu(h)v$.
4. it is finitely generated, which implies (see Proposition below) that the principal grading on $M$ is upper bounded, and the graded components are finite-dimensional.

**Remark 97.**

There are many versions of category $\mathcal{O}$ if we modify (4), but we'll work with essentially the "smallest" version, so that our category is not too big.

**Definition 98** (principal grading on $\widehat{\mathfrak{g}}$)**.**

The principal grading on $\widehat{\mathfrak{g}}$ is given by $\operatorname{ad} \widehat{h} := \frac{1}{2} \operatorname{ad} h + (m+1) \operatorname{ad} d$, where $[\widehat{h}, e_i] = e_i$, and $m$ is the height of the maximal root $\theta$ (e.g., if $\theta = \sum_\Delta n_i \alpha_i$, then $m := \sum n_i$). (Here $h$ is the element in $\mathfrak{h}$ such that $[h, e_i] = 2e_i$.)

**Proposition 99.**

Let $M \in \mathcal{O}$. Then $M = \bigoplus_{n\in\mathbb{Z}} M_n$, where $\widehat{h}|_{M_n} = n \cdot \mathrm{id}$. Furthermore, there exists $N \in \mathbb{Z}$ such that for all $n > N$, $M_n = 0$.

***Proof.***
Let $v_1, \ldots, v_k$ be the generators of $M$. Then there exists $N_1$ such that $e_{i_1} \cdots e_{i_{N_1}} v_1 = 0$. Suppose $N_2$ is the biggest degree of $v_i$ with respect to $\widehat{h}$. Then $N = N_1 + N_2$ satisfies this condition.

□

$$M = \bigoplus M_k$$

Note that $c|_{M_k} = k \cdot \mathrm{id}$.
Roughly speaking, $\mathcal{O} = \bigoplus_k \mathcal{O}_k$, where $\mathcal{O}_k$ is the category of representations of the level $k$.

For finite-dimensional $\mathfrak{g}$, category $\mathcal{O}$ enjoys some nice properties:

- all objects have finite length.
- it contains finite-dimensional $\mathfrak{g}$-modules.
  Unfortunately, **both** of these properties fail for $\widehat{\mathfrak{g}}$. The second property has a simple example: set $k = 0$, then consider the evaluation representation $\mathfrak{g}[z, z^{-1}] \to \mathfrak{g}$ by sending $z \mapsto ? \in \mathbb{C}$, specializing $z$ to some complex number (say $1$). Then $\mathfrak{g} \to \mathrm{End}(V)$ for some finite-dimensional $V$, which fails the condition that $\widehat{\mathfrak{n}}_+$ acts locally nilpotently. As a concrete example, consider $\mathfrak{g} = \mathfrak{sl}_2$ and $V = \mathbb{C}^2$ the tautological representation. Then $\ldots fefefefefe \cdot v \neq 0$ so long as $ev \neq 0$.

The point of all this is that finite-dimensional modules are not the correct objects to study. Instead it'll be something called **integrable modules**.

**Definition 100** (integrable module)**.**

A $\widehat{\mathfrak{g}}$-module in category $\mathcal{O}$ is **integrable** if it is integrable with respect to all copies of $\mathfrak{sl}_2$ generated by $\langle e_i, f_i, h_i \rangle$ (i.e., $f_i$ act locally nilpotently). (Recall that a representation of $\mathfrak{sl}_2$ is **integrable** if it is a (possibly infinite) direct sum of finite-dimensional $\mathfrak{sl}_2$-modules; however, note that these copies of $\mathfrak{sl}_2$ intertwine in such a way that as a $\widehat{\mathfrak{g}}$-module, the representation **does not** decompose as a direct sum of finite-dimensional $\widehat{\mathfrak{g}}$-modules.)

**Example 101.**

For $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_2$, we are looking for integrable quotient of a **Verma module**

$$M(\lambda, k) = \underbrace{\mathrm{Ind}_{\widehat{\mathfrak{h}}\oplus\widehat{\mathfrak{n}}_+}^{\widehat{\mathfrak{g}}} \mathbb{C}_{\lambda,k}}_{c|_{\mathbb{C}_{\lambda,k}}=k\cdot\mathrm{id}, \quad \mathfrak{h}\ni h|_{\mathbb{C}_{\lambda,k}}=\lambda(h)} = \mathcal{U}(\widehat{\mathfrak{n}}_-)v_{\lambda,k}.$$

Now we want this to have an integrable quotient with respect to both generating $\mathfrak{sl}_2$-triples, $\langle e_0, f_0, h_0\rangle$ and $\langle e_1, f_1, h_1\rangle$. The latter set is the "usual" set and the condition that it's integrable with respect to this set implies that $\lambda \in \mathbb{Z}_{\geq 0}$; the integrability on the other set shows that $h_0 v_{\lambda,k} = \underbrace{(k-\lambda)}_{\geq 0} v_{\lambda,k}$, so $k - \lambda \in \mathbb{Z}_{\geq 0}$.

In particular, there is a unique integrable module of level 0 (namely the trivial module), and two irreducible integrable modules of level 1, corresponding to $\lambda = 0, 1$.

# Oct 18

## Sugawara construction for $\widehat{\mathfrak{sl}_2}$ and applications to category $\mathcal{O}$

### Invariant forms on $\mathfrak{g}[z, z^{-1}]$

Let $\langle , \rangle$ be an invariant scalar product on $\mathfrak{g}$. Then we can extend in the obvious way to an invariant scalar product $\langle x(z), y(z)\rangle \in \mathbb{C}[z, z^{-1}]$. Then we can multiply by any differential form $F(z)\,\mathrm{d}z$, and we obtain a form

$$x(z), y(z) \mapsto \mathrm{Res}_{z=0}\langle x(z), y(z)\rangle F(z)\,\mathrm{d}z \in \mathbb{C}.$$

So this suggests the construction of some element in the universal enveloping algebra, i.e. the "Casimir" element in $\mathcal{U}(\mathfrak{g}[z, z^{-1}])$.

When $F(z) = z^{-n-1}$, then $\langle x[r], y[s]\rangle_n = \delta_{r+s=n}\langle x, y\rangle$. For any basis $\{x_a\}$ of $\mathfrak{g}$ and $\{x^a\}$ the dual basis, the form $\langle , \rangle_n$ suggests the element

$$\sum_{r+s=n} \sum_{\alpha} x_\alpha[r] x^\alpha[s],$$

which is **not** an element of the universal enveloping algebra, but **is** an element of $\mathcal{U}(\widetilde{\mathfrak{g}[z, z^{-1}]}) = \lim_{\infty\leftarrow N} \mathcal{U}(\mathfrak{g}[z, z^{-1}])/\mathcal{U}(\mathfrak{g}[z, z^{-1}])z^N\mathfrak{g}$, and has a well-defined action on category $\mathcal{O}$.

**Lemma 102.**

$\sum_\alpha x_\alpha[r] x^\alpha[s] = \sum_\alpha x^\alpha[s] x_\alpha[r]$.

*Proof.*
The difference is an element of $\mathfrak{g}[r+s]$ which commutes with $\mathfrak{g}[0]$. But since $\mathfrak{g}$ is simple, there are no such elements.

□

This implies that we can write

$$S_n := \sum_{r+s=n,} \sum_{s>0} \sum_{\alpha} x_\alpha[r]x^\alpha[s] + \sum_{r+s=n,} \sum_{s\le 0} \sum_{\alpha} x^\alpha[s]x_\alpha[r].$$

There is one huge problem: this element is **not central**.

**Example 103.**

Let $\mathfrak{g} = \mathfrak{sl}_2$. We'll show that $S_0$ as constructed is not central. Then

$$S_0 = e[0]f[0] + f[0]e[0] + \frac{1}{2}h[0]^2 + 2\sum_{r>0}\left(e[-r]f[r] + f[r]e[r] + \frac{1}{2}h[-r]h[r]\right).$$

We can check commutativity on the generators of $\widehat{\mathfrak{sl}_2}$. First, $[S_0, x[0]] = 0$, so that is not a problem. But

$$\begin{aligned}[S_0, h[1]] = &-2e[1]f[1] + 2e[0]f[1] + ef[1]e[0] - 2f[0]e[1] \\ &\underbrace{+ 2\sum_{r>0} -2e[-r+1]f[r] + 2e[-r]f[r+1] + 2f[-r+1]e[r] - 2f[-r]e[r+1]}_{2(-2e[0]f[1]+2f[0]e[1])} \\ = &-4h[1].\end{aligned}$$

So $[S_0, h[1]] = -4h[1]$, which is nonzero. We also know that $[S_0, x[m]] \in \mathfrak{sl}_2[z, z^{-1}]$, i.e. $\operatorname{ad} S_0 : \mathfrak{sl}_2[z, z^{-1}] \to \mathfrak{sl}_2[z, z^{-1}]$ is an $\widehat{\mathfrak{sl}}_2$-invariant derivation. But we already checked this before — then it must come the Witt algebra, and since it is level 0, it must be proportional to $z\partial_z$.

**Proposition 104.**

$[S_0, -] = -4z\partial_z$. Similarly, we can define $[S_n, -] = -4z^{n+1}\partial_z$.

**Corollary 105.**

The elements $\{S_n \mid n \in \mathbb{Z}\}$ generates a central extension of the Witt algebra $W \subset \mathcal{U}(\widetilde{\mathfrak{g}[z, z^{-1}]})$. (Note that the center of the $\mathcal{U}(\widetilde{\mathfrak{g}[z, z^{-1}]})$ is trivial, although we have not showed this before.) This implies that $[[S_n, S_m], -]$ generates a copy of $Vir$, the Virasoro algebra, since $[S_n, S_m] = S_{n+m} + ?$ and thus gives us the central extension.

For $\widehat{\mathfrak{sl}_2}$ we have $[S_0, h[1]] = -2(c+2)h[1]$.

**Proposition 106.**

$[S_0, -] = -2(c+2)z\partial_z =: -2(c+2)[d, -]$. More generally, $[S_n, -] = -2(c+2)z^{n+1}\partial_z$. Therefore we have an embedding $Vir \hookrightarrow \widetilde{\mathcal{U}(\widehat{\mathfrak{sl}_2})}]$.

**Remark 107.**

The level $c = -2$ is special, and called **critical**. This is because all $S_n$ become central in $\widetilde{\mathcal{U}(\widehat{\mathfrak{sl}_2})}/(c+2)$.

**Theorem 108** (Casimir elememt)**.**

$\widehat{C} := S_0 + 2(c+2)d \in \widetilde{\mathcal{U}(\widehat{\mathfrak{sl}_2})}$ is central. We call this the **Casimir element**.

## Category $\mathcal{O}$

**Definition 109** (Verma module)**.**

Let $k \in \mathbb{C}$. Then $\mathcal{O}_k \ni M(\lambda, k)$ the **Verma module** generated by $v_{\lambda,k}$ subject to conditions $\widehat{\mathfrak{n}}_+ v_{\lambda,k} = 0$, $h[0]v_{\lambda,k} = \lambda v_{\lambda,k}$, and $cv_{\lambda,k} = kv_{\lambda,k}$.

**Definition 110** (highest weight module)**.**

$V$ is a **highest weight module** of level $k$ if $V = M(\lambda, k)/-$ for some $\lambda$.

**Proposition 111.**

Any module $M$ from $\mathcal{O}_k$ has a finite filtration $M \supset M_0 \supset \cdots \supset M_N$ such that $M_i/M_{i+1}$ is highest weight for each $i$.

**Theorem 112.**

Let $\mathbb{Q} \ni k < -2$ or $k \notin \mathbb{Q}$. Then any object of $\mathcal{O}_k$ has finite length.

***Proof.***
It's sufficient to do this for Verma modules $M(\lambda, k)$. Then we just need to show that there are finitely many singular vectors. The reason is that if there is a singular vector, then the eigenvalue for Casimir is the highest weight.

**Lemma 113.**

$\widehat{C}|_{M(\lambda,k)}$ is constant.

Then when you go down the Verma module, the eigenvalue of the second term $2(c+2)d$ in the Casimir element increases, while if we have a singular vector then the value of the first term $S_0$ is always just the value of the Casimir on the highest vector for $\widetilde{\mathfrak{sl}_2}$. So these are bounded below and there are finitely many weight spaces. We'll finish the proof next class.

□

# Oct 20

## Generalities about category $\mathcal{O}$ (in the infinite-dimensional setting)

**Theorem 114.**

Let $\mathbb{C} \ni k < -2$ (where $a < b \iff b - a \in \mathbb{Q}_{>0}$). Modules from $\mathcal{O}_k$ have finite length.

***Proof.*** First, it's sufficient to show that Verma modules $M(\lambda, k)$ have finite length; this is because every module from category $\mathcal{O}$ can be filtered in such a way that all subsequent quotients are highest weight modules (i.e., quotients of Verma modules).

Next, suppose that some Verma module $M(\lambda, k)$ is infinite length, i.e. that for all $N$, there exists a chain of strict inclusions $M(\lambda, k) \supsetneq M_1 \supsetneq M_2 \supsetneq \cdots \supsetneq M_N$. Consider the weight decomposition $M(\lambda, k) = \mathcal{U}(\widehat{\mathfrak{n}}_-) \cdot v_{\lambda,k}$: we define

$$M(\lambda, k)_{\mu,m} := \{v \in M(\lambda, k) \mid h[0]v = \mu v, \quad dv = mv\}.$$

To do this, we need to make some choice, to determine the action $dv_{\lambda,k}$. In this case we declare $dv_{\lambda,k} = 0$. From this we see that

$$d\left(\prod e[r_i]f[s_i]h[p_i]v_{\lambda,k}\right) = \sum(r_i + s_i + p_i)(-),$$

and thus that the weight spaces are finite-dimensional. So the sequence of inclusions also induces a sequence of inclusions on each weight space.

Now consider the quotients $M_i/M_{i+1}$, and choose some highest vector (not necessarily generating the quotient) $v_i$ annihilated by $\widehat{\mathfrak{n}}_+$, which always exists. Lift this $v_i$ to some $\widetilde{v_i} \in M(\lambda, k)$. All of the $\widetilde{v_i}$ are linearly independent. This means that there exist infinitely many pairs $(\mu, m)$ such that $(M_i/M_{i+1})_{\mu,m} \ni v_i$ where $\widehat{\mathfrak{n}}_+ v_i = 0$. But let us consider the possible values for $\mu, m$. First, $m \leq 0$, and $\mu \in \lambda + 2\mathbb{Z}$. If $m = 0$ then $\mu \in \lambda - 2\mathbb{Z}_{\geq 0}$.

Now we need the following lemma. Let $\widehat{C} = S_0 + 2(k+2)d$ be the Casimir element. On the Verma module $M(\lambda, k)$, it acts by the scalar $\frac{\lambda(\lambda+2)}{2}$. The proof is that it acts by this scalar on the highest vector $v_{\lambda,k}$, and it commutes with everything with $\mathcal{U}(\widehat{\mathfrak{n}}_-)$ which also generates everything in $M(\lambda, k)$.

Next, we also need another lemma. Consider the action of $\widehat{C} \cdot v_i$ of some highest vector in a quotient. Then $\widehat{C} v_i = \left(\frac{\mu(\mu+2)}{2} + 2(k+2)m\right) v_i$. The proof is postponed to next time.

These two prove the theorem for rational $k$. □

# General setting

Now let's discuss category $\mathcal{O}$ in general. Let $\mathcal{L}$ be a (possibly infinite-dimensional) Lie algebra which is $\mathbb{Z}$-graded, i.e. $\mathcal{L} = \bigoplus_{n \in \mathbb{Z}} \mathcal{L}_n$, such that $\mathcal{L}_0$ is abelian, and $\dim \mathcal{L}_n < \infty$ for all $n$. This applies to all Lie algebras we have seen so far, including the Heisenberg algebra $\mathfrak{a}$, the Virasoro algebra $Vir$, and the Kac-Moody algebras $\widehat{\mathfrak{g}}$. In this general situation we can always define a **Verma module** $M(\lambda) := \mathcal{U}(\mathcal{L}_{<0}) v_\lambda$ corresponding to any character $\lambda \in \mathcal{L}_0^*$. This module is still $\mathbb{Z}$-graded (only negative/nonpositive components), and the dimensions of the graded components are finite. So we can do the usual business with category $\mathcal{O}$.

Let's describe all simples in $\mathcal{O}$. In the usual case (semisimple Lie algebras), a simple module is a quotient of a Verma module, and there is a minimal quotient. Here, any proper submodule of $M(\lambda)$ is contained in $M(\lambda)_{<0}$, the strictly negative part of $M(\lambda)$, hence the sum of all proper submodules is still a proper submodule (contained in $M(\lambda)_{<0}$).

**Proposition 115.**

There exists a unique maximal proper submodule $N(\lambda) \subset M(\lambda)$. Thus, all simple modules in $\mathcal{O}$ are of the form $M(\lambda)/N(\lambda) =: L(\lambda)$.

The main problem is to find the Poincare series/characters of $L(\lambda)$ with respect to $\mathcal{L}_0$ and $d$. What we expect is that 1) for $\widehat{\mathfrak{g}}$, for integrable $L(\lambda)$ there will be some analogue of Weyl character formula, such that Jacobi and Macdonald identities are some particular cases, and 2) (we will continue with this next time) something even more interesting for $Vir$. The problem with $Vir$ is that it is not a Kac-Moody algebra, and it is not determined by a Cartan matrix, so much less is known here. But what we have seen is that any affine Kac-Moody algebra contains some $Vir$, so representations of it arise everywhere, making it "universal" in some sense. So it is really important. So we cannot say "integrable" for $Vir$ because there is no Lie group, but we can distinguish some class of modules (called minimal Virasoro modules) which share many properties with integrable modules.

# Oct 23

## Category-$\mathcal{O}$ representations of $\widehat{\mathfrak{sl}_2}$

In fact, most of what we say today is true more generally for $\widehat{\mathfrak{g}}$, but it's easier to first just consider the case of $\mathfrak{g} = \mathfrak{sl}_2$.

First, we need to finish the finite-length property for category-$\mathcal{O}_k$ modules for $\{k \in \mathbb{Q}, \quad k+2<0\}$ or $\{k \in \mathbb{C} \setminus \mathbb{Q}\}$ (recall Theorem 112). We reduced this to the study of Verma modules $M(\lambda, k)$. What we are checking is that there are finitely many possibilities for the highest vector in $M(\lambda, k)$. The highest vector has to satisfy the property that its eigenvalue under the Casimir is the same. So there exists a highest vector of weight $(\mu, -n)$ (where $\mu$ is the eigenvalue of $h[0]$ and $-n$ is the eigenvalue of $d$) in some subquotient of $M$, for $\mu = \lambda + 2m$ and $m \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 0}$. We were checking that there were finitely many possibilities for $(\mu, -n)$ to have the same eigenvalue under the Casimir. Now the eigenvalue of the Casimir $S_0 + 2(k+2)d$ is $\frac{\lambda(\lambda+2)}{2}$, while the eigenvalue on the highest vector (of weight $(\mu, -n)$) is $\frac{\mu(\mu+2)}{2} - 2(k+2)n$. Setting these equal and writing $\mu = \lambda + 2m$, we have

$$\begin{aligned} \lambda + 2\lambda^2 &= \lambda^2 + 2\lambda + 4m\lambda + 4m^2 + 2m - 4(k+2)n, \\ \implies 0 &= 4m\lambda + 4m^2 + 2m - 4(k+2)n. \end{aligned}$$

Here $\lambda$ and $k+2$ are fixed. So we have two cases: first, in the $m-n$ plane, the equation is a parabola, and we have finitely many integer points with $n \geq 0$. The second case is if $k+2 \notin \mathbb{Q}$. Let's split $\mathbb{C} = \mathbb{Q} \oplus \mathbb{Q}^\perp$ as vector spaces (yes… this is weird). Then $k+2 = k_1 + k_2$, and $\lambda = \lambda_1 + \lambda_2$, with $k_1, \lambda_1 \in \mathbb{Q}$ and $k_2, \lambda_2 \in \mathbb{Q}^\perp$. Then we find that

$$4m\lambda_2 = 4nk_2.$$

Since both of these are nonzero, we deduce that $m$ should be proportional to $n$. But then the line $m = cn$ for some $c$, intersects the parabola at finitely many points (no matter which way the parabola is facing). So either way, we see that there are only finitely many possibilities.

## Characters

Suppose $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra, and $V$ is a finite-dimensional $\mathfrak{g}$-module. Recall that if $V$ is finite-dimensional, then $V$ is integrable, i.e. it integrates to a $G$-representation, where $G$ is the connected, simply-connected Lie group corresponding to $\mathfrak{g}$. Let $\pi : G \to \mathsf{GL}(V)$ be this homomorphism. Then we define $\mathrm{ch}_V(g) := \mathrm{Tr}_V(\pi(g))$. This has the usual property that it's conjugation-invariant: $\mathrm{ch}_V(g) = \mathrm{ch}_V(hgh^{-1})$ for all $h, g \in G$. So this defines a function on conjugacy classes of $G$. On the other hand, almost all conjugacy classes (i.e., complement of union of these conjugacy classes forms a set of measure $0$, or alternatively viewed as an algebraic group, the complement is a proper Zariski-closed subset) have a representative in the maximal torus (if $G$ is compact, it's true for every conjugacy class). In fact, the representatives in the maximal torus form a single $W$-orbit, where $W = N_G(T)/T$ is the Weyl group. So this means that the character is uniquely determined by its values on the maximal torus.

**Proposition 116.**

$\mathrm{ch}_V(g)$ is unique determined by the values $\mathrm{ch}_V(t)$ for $t \in T$, and furthermore $\mathrm{ch}_V|_T \in \mathbb{C}[T]^W$.

We want something similar in the infinite-dimensional case. The question is what to do, because we don't have the Lie group, so we have an issue trying to define the character as a trace of the element in the Lie group. The idea is that we still have a notion of a function on the torus (though maybe not the whole "group").

Let us return to the finite-dimensional case. Suppose that we have the decomposition $V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu$, of $V$ into $\mathfrak{h}$-eigenspaces, so that for $h \in \mathfrak{h}$, we have $h|_{V_\mu} = \mu(h) \cdot \mathrm{id}$. Then (essentially by definition), we have

**Proposition 117.**

$\mathrm{ch}_V(t) = \sum_{\mu \in \mathfrak{h}^*} t^\mu \cdot \dim V_\mu$.

Let's regard $t^\mu$ as a formal symbol, satisfying that $t^{\mu_1} \cdot t^{\mu_2} = t^{\mu_1 + \mu_2}$.

This means that to determine the character, it suffices to understand the decomposition of $V$ into $\mathfrak{h}$-weight spaces, which is indeed easy to generalize to the infinite-dimensional case.

Suppose

$$M = \bigoplus M_\mu, \quad \dim M_\mu < \infty.$$

Then we can write down the character $\mathrm{ch}_M(t)$ as an infinite sum, but it will still be well-defined.

In particular, we can write down the character of a Verma module (in the finite-dimensional case).

**Example 118** (character of a Verma module)**.**

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra and $M = M(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b}_+)} \mathbb{C}\lambda$ be the Verma module, where $\mathfrak{b}_+ = \mathfrak{n}_+ \oplus \mathfrak{h}$. Here, $\mathfrak{n}_+$ acts on $\mathbb{C}\lambda$ by $0$, while $\mathfrak{h}$ acts on $\mathbb{C}\lambda$ by $\lambda : \mathfrak{h} \to \mathbb{C}$. Then by PBW, this module is a free $\mathcal{U}(\mathfrak{n}_-)$-module generated by a highest-weight vector $v_\lambda$. So the character is given by

$$\mathrm{ch}_{M(\lambda)}(t) = t^\lambda \cdot \mathrm{ch}_{\mathcal{U}(\mathfrak{n}_-)}(t) \overset{\text{PBW}}{=} t^\lambda \cdot \mathrm{ch}_{S(\mathfrak{n}_-)}(t).$$

To compute this, we just need to know the action of the torus on the space of generators (namely $\mathfrak{n}_-$ ). Let $\Delta_+$ denote the positive roots. Recall that $\mathfrak{n}_- = \bigoplus_{\alpha \in \Delta_+} f_\alpha$, and each $f_\alpha$ gives rise to $t^{-\alpha}$. So we get that

$$\begin{aligned} \mathrm{ch}_{M(\lambda)}(t) &= t^\lambda \cdot \mathrm{ch}_{S(\mathfrak{n}_-)}(t), \\ &= t^\lambda \cdot \mathrm{ch}_{\bigotimes_{\alpha \in \Delta_+} \mathbb{C}[f_\alpha]}(t), \\ &= t^\lambda \cdot \prod_{\alpha \in \Delta_+} \mathrm{ch}_{\mathbb{C}[f_\alpha]}(t), \\ &= t^\lambda \cdot \prod_{\alpha \in \Delta_+} \frac{1}{1 - t^{-\alpha}}. \end{aligned}$$

Now, recall that Verma modules span the Grothendieck group of the category $\mathcal{O}$, so we can compute the characters of all of the modules as linear combinations of the characters of the Verma modules (as computed here).

Now consider (possibly infinite-dimensional) Lie algebra $\mathcal{L}$ with the usual assumptions: $\mathbb{Z}$-graded, so that $\mathcal{L} = \bigoplus_{n \in \mathbb{Z}} \mathcal{L}_n$, and $\mathcal{L}_0$ is abelian. Then we can define the character $\mathrm{ch}_M(t, q)$ of any graded $\mathcal{L}$-module $M$ such that $M = \bigoplus_{n \in \mathbb{Z}} M_n$ and $M_n = \bigoplus_{\mu \in L_0^*} M_{\mu,n}$, $\mathcal{L}_0$ acts semisimply, and the joint-eigenspaces $M_{\mu,n}$ are finite-dimensional. The variable $t$ comes from the "torus" $\mathcal{L}_0$ action (i.e. $\mu$), and the $q$ comes from the $\mathbb{Z}$-grading (i.e. $n$). Thus we define:

**Definition 119** (character)**.**

Let $\mathcal{L}$ and $M$ be as above, i.e., $\mathcal{L}$ is $\mathbb{Z}$-graded with $\mathcal{L}_0$ abelian, $M$ is $\mathbb{Z}$-graded and $\mathcal{L}_0$ acts semisimply, with finite-dimensional joint-eigenspaces. Then we define

$$\mathrm{ch}_M(t, q) := \sum_{\substack{\mu \in \mathcal{L}_0^* \\ n \in \mathbb{Z}}} (\dim M_{\mu,n}) \cdot t^\mu q^n.$$

In other words, to define this, we consider an extended Lie algebra

$$0 \to \mathcal{L} \to \widetilde{\mathcal{L}} \to \mathbb{C}d \to 0,$$

where the adjoint action of $d$ gives the $\mathbb{Z}$-grading, and we consider $\widetilde{\mathcal{L}}$-modules (i.e., graded modules over $\mathcal{L}$). We have a bigger abelian algebra $\widetilde{\mathcal{L}_0} = \mathbb{C}d \oplus \mathcal{L}_0$, and we decompose the module with respect to the action of $\widetilde{\mathcal{L}_0}$, and require that the weight spaces of $\widetilde{\mathcal{L}_0}$ (which are just the joint-weight spaces of $\mathcal{L}_0$) should be finite-dimensional.

**Example 120.**

Let $\widehat{\mathfrak{g}} = \mathbb{C}c \oplus \mathfrak{g}[z, z^{-1}]$ be a nontrivial central extension. Let $\widetilde{\mathfrak{g}} = \mathbb{C}d \oplus \widehat{\mathfrak{g}}$.

**Proposition 121.**

Any category-$\mathcal{O}$ module $M \in \mathcal{O}(\widetilde{\mathfrak{g}})$ has a well-defined character $\mathrm{ch}_M(t, q)$. Namely, let us define

$$M_{\mu,n} = \{v \in M \mid h[0]v = \mu(h)v \, \forall h \in \mathfrak{h}, \quad dv = -nv\}.$$

We'll define the character by:

$$\mathrm{ch}_M(t, q) = \sum (\dim M_{\mu,n}) \cdot t^\mu q^n.$$

***Proof.***

It's sufficient to check this for Verma modules $M(\lambda) = \mathcal{U}(\widetilde{\mathfrak{g}}) \otimes_{\mathcal{U}(\mathfrak{h}+\mathbb{C}c+\mathbb{C}d+\mathfrak{n}_+)} \mathbb{C}\lambda$. As a vector space with an action of the extended Cartan algebra $\widetilde{\mathfrak{h}} = \mathfrak{h} + \mathbb{C}c + \mathbb{C}d$, this is just $\mathcal{U}(\widehat{\mathfrak{n}}_-) \cdot v_{\lambda,k}$ (where $k$ is the eigenvalue of the central element $c$). We just need to show that the weight spaces in $\mathcal{U}(\widehat{\mathfrak{n}}_-)$ have finite dimension. But this is indeed the case, since $\widehat{\mathfrak{n}}_-$ is negatively graded with respect to some linear combination of the multi-grading. (In other words, we can take some element from $\widetilde{\mathfrak{h}}$ such that all eigenvalues with respect to this element are strictly negative integers, and this means that all of the graded components with respect to this particular element are already finite-dimensional, and hence all of the weight spaces in $\mathcal{U}(\widehat{\mathfrak{n}}_-)$ are finite-dimensional.

□

In fact, we can even compute this character. It will be

$$\mathrm{ch}_{M(\lambda)}(t, q) = t^\lambda \cdot \prod_{\alpha \in \Delta_+} \frac{1}{\prod_{n=0}^{\infty}(1 - q^n t^{-\alpha})} \prod_{\alpha \in \Delta_+} \frac{1}{\prod_{n=1}^{\infty}(1 - q^n t^{\alpha})} \prod_{h_i \text{ basis for } \mathfrak{h}} \frac{1}{\prod_{n=1}^{\infty}(1 - q^n)}.$$

The first product corresponds to elements of the form $f_\alpha \cdot z^?$, the second to $e_\alpha \cdot z^?$, and the third to $h_i \cdot t^?$.

# Oct 25

## $\mathcal{O}$ for $\widehat{\mathfrak{sl}_2}$ (then generalizing to $\widehat{\mathfrak{g}}$)

### Verma module $M(\lambda, k)$

Recall that $M(\lambda,k) = \mathcal{U}(\widehat{\mathfrak{g}}) \otimes_{\mathcal{U}(\widehat{\mathfrak{b}}_+)} \mathbb{C}_{\lambda,k} = \mathcal{U}(\widetilde{\mathfrak{g}}) \otimes_{\mathcal{U}(\widetilde{\mathfrak{b}}_+)} \mathbb{C}_{\lambda,k,0}$, where $\widehat{\mathfrak{b}}_+ = \widehat{\mathfrak{n}}_+ + \widehat{\mathfrak{h}}$, $\mathbb{C}_{\lambda,k} = \mathbb{C}_{\lambda,k,0}$ is the one-dimensional $\widehat{\mathfrak{b}}_+$-representation where $\widehat{\mathfrak{n}}_+$ acts by $0$ and $h \in \mathfrak{h}$ acts by $\lambda(h)$, $c$ acts by $k$, and $d$ acts by $0$. (Recall that $\widetilde{\mathfrak{b}}_+ = \widehat{\mathfrak{b}}_+ + \mathbb{C}d$.) As a vector space,

$$\mathcal{U}(\widetilde{\mathfrak{g}}) \otimes_{\mathcal{U}(\widetilde{\mathfrak{b}}_+)} \mathbb{C}_{\lambda,k,0} = \mathcal{U}(\widehat{\mathfrak{n}}_-) \cdot v_{\lambda,k},$$

where $\widehat{\mathfrak{n}}_- = \mathfrak{n}_- \oplus z^{-1}\mathfrak{g}[z^{-1}]$. Now $\widehat{\mathfrak{n}}_-$ has a natural basis coming from the root basis in $\mathfrak{n}_-, \mathfrak{h}, \mathfrak{n}_+$ (although we will need to choose some particular basis of $\mathfrak{h}$ but this doesn't really matter). Let

$$\left\{ \underbrace{e_\alpha \in \mathfrak{n}_+}_{\text{basis of } \mathfrak{n}_+} \mid \alpha \in \underbrace{\Phi_+}_{\text{positive roots}} \right\}, \quad \left\{ \underbrace{f_\alpha \in \mathfrak{n}_-}_{\text{basis of } \mathfrak{n}_-} \mid \alpha \in \underbrace{\Phi_+}_{\text{positive roots}} \right\}, \quad \left\{ \underbrace{h_i \in \mathfrak{h}}_{\substack{\text{basis of } \mathfrak{h}, \\ \text{correspond to simple roots}}} \mid h_i = [e_i, f_i] \right\}.$$

Then we have a basis of $\widehat{\mathfrak{n}}_-$ given by $f_\alpha[r]$ for $r \leq 0$ and $e_\alpha[r], h_i[r]$ for $r < 0$.

The **weight decomposition of** $M(\lambda, k)$ is given by

$$M(\lambda,k) = \bigoplus_{\underbrace{\mu}_{\text{weight}}, \underbrace{n}_{\in \mathbb{Z}_{\geq 0}}} M(\lambda,k)_{\mu,-n} \qquad h \to \mu(h), \quad d \to -n.$$

The **character** is

$$\begin{aligned}
\operatorname{ch}^{t,q}_{M(\lambda,k)} &= \sum_{\mu,n} t^\mu q^n \dim M(\lambda,k)_{\mu,-n}, \\
&= t^\lambda \cdot \operatorname{ch}_{\mathcal{U}(\widehat{\mathfrak{n}}_-)}, \\
&= t^\lambda \cdot \operatorname{ch}_{S(\widehat{\mathfrak{n}}_-)}, \\
&= t^\lambda \cdot \operatorname{ch}_{\mathbb{C}[f_\alpha[r+1], e_\alpha[r], h_\alpha[r]]_{r<0}}, \\
&= t^\lambda \cdot \prod_{\alpha\in\Phi^+} \prod_{n\geq 0} \frac{1}{1-t^{-\alpha}q^n} \prod_{\alpha\in\Phi^+} \prod_{n>0} \frac{1}{1-t^{\alpha}q^n} \left( \prod_{n>0} \frac{1}{1-q^n} \right)^{\operatorname{rk}\mathfrak{g} = \dim \mathfrak{h}}.
\end{aligned}$$

In category $\mathcal{O}$, you want to compute the character of any irreducible object. But in the Grothendieck group ($K$-group), the classes of Verma modules form a basis, so the class of any object can be expressed as a linear combination of the classes of Verma modules. The difference between the finite-dimensional semisimple Lie algebras and the infinite-dimensional Lie algebras is that in the infinite-dimensional situation, it may happen that we have some infinite linear combination (the length of the filtration may not be finite).

## Some examples of $L(\lambda, k) = M(\lambda, k)/N(\lambda, k)$

Recall that $N(\lambda, k)$ is a maximal proper submodule.

**Proposition 122.**

For Weil generic $(\lambda, k)$, then $M(\lambda, k)$ is irreducible.

For $\mathfrak{g}$ finite-dimensional semisimple, the condition is $\langle \alpha^\vee, \lambda\rangle = \lambda(h_\alpha) \notin \mathbb{Z}_{\geq 0}$ for all $\alpha \in \Phi^+$, which is a finite number of closed conditions. For us, this will be an infinite number of conditions, so we need $\lambda$ outside of a countable union of proper Zariski closed subsets in $\mathfrak{h}^*$, which is called "Weil generic."

***Proof.*** $M(\lambda, k)$ contains a proper submodule $N$ iff there exists a singular vector $v \in M(\lambda, k)_{\mu,-n}$ which is **not** a highest weight vector (i.e. $\mu \neq \lambda$ or $k \neq 0$) such that $\widehat{\mathfrak{n}}_+ v = 0$; this is a Zariski closed conditions on $\lambda, k$. There are countably many pairs $(\mu, -n)$. So the condition that $M(\lambda, k)$ is reducible is the union of countably of many Zariski closed conditions on $(\lambda, k)$.

So we need to check that for all $\mu, -n$ there exists $(\lambda, k)$ where there is no such $v$ in $M(\lambda, k)_{\mu,-n}$. (This way the union is not everything.)

One way to prove it is to use the Casimir operator $\widehat{C} = S_0 + 2(c + h^\vee)d$ where $h^\vee$ is just some number. The eigenvalues on $v_{\lambda,k}$ and on $v \in M(\lambda, k)_{\mu,-n}$ where $\widehat{\mathfrak{n}}_+ v = 0$ are different. But on any Verma module Casimir commutes with every element, so Casimir acts by a constant, and its eigenvalue should be always the eigenvalue on $v_{\lambda,k}$. However if we have a singular vector then it will act by something else; we can compute this. For example, for $\widehat{\mathfrak{sl}_2}$ then Casimir acts on highest weight vector by $\frac{\lambda(\lambda+2)}{2}$ while on any singular vector it acts by $\frac{\mu(\mu+2)}{2} - 2(k+2)n$. Since these must be equal, we have a very explicit closed condition on the singular vectors.

Another way (which is even more general, and you can apply it even when you do not have a Casimir, such as the Virasoro algebra) is the following. Suppose $|k| \gg 0$. We have a basis $e_\alpha[r]$, $f_\alpha[r]$, and $h_i[r]$ for $r \in \mathbb{Z}$. Then $[e_\alpha[r], f_\alpha[s]] = h_\alpha[r+s] + r\delta_{r+s=0}$. Change the generators by dividing by $\sqrt{k}$, i.e. $\widetilde{e_\alpha}[r] := \frac{e_\alpha[r]}{\sqrt{k}}$. Then we get relations

$$[\widetilde{e_\alpha}[r], \widetilde{f_\alpha}[s]] = \frac{1}{\sqrt{k}}\widetilde{h_\alpha}[r+s] + r\delta_{r+s=0}.$$

So these "almost commute" and $\mathcal{U}(\widehat{\mathfrak{g}})/(c-h)$ can be regarded as a deformation of the Heisenberg algebra $\mathfrak{a}_\mathfrak{g}$, defined by the same generators $\widetilde{e_\alpha}[r], \widetilde{f_\alpha}[r], h_i[k]$, (indeed for any $\mathfrak{g} \ni x \rightsquigarrow \widetilde{x}[r]$ we have a mapping of each element to its corresponding new generator) and new relations $[\widetilde{x[r]}, \widetilde{y[r]}] = r\delta_{r+s=0}\langle x, y\rangle$.

So this algebra is just the direct sum of $\dim \mathfrak{g}$ copies of the usual $\mathfrak{a}$: we have

$$\mathcal{U}(\mathfrak{a}_{\mathfrak{g}}) = \mathfrak{a}^{\oplus \dim \mathfrak{g}},$$

and $(\sqrt{k})^{-1}$ is the **deformation parameter**.

Then $M(\lambda, k)$ is a deformation of $F^{\otimes \dim \mathfrak{g}} \otimes \mathbb{C}[f_\alpha[0]]_{\alpha \in \Phi^+}$. This is irreducible. This means that there are no singular vectors with $n > 0$ in the limit $k \to \infty$ (note that this is independent of $\lambda$). The space of singular vectors is $v_{\text{highest}}^{\otimes \dim \mathfrak{g}} \otimes \mathbb{C}[f_\alpha]_{\alpha \in \Phi^+}$.

Now we have $\bigoplus M(\lambda, k)_{\mu,0} = \mathcal{U}(\mathfrak{n}_-) v_{\lambda,k} \simeq M(\lambda)$ for $\mathfrak{g}$, irreducible for Weil generic $\lambda$. (In fact you can replace condition $|k| \gg 0$ with $|\lambda + k| \gg 0$ which gives a Heisenberg algebra without a center, but we end up still getting an irreducible module.)

□

# Oct 27

## Shapovalov form (contravariant form) on Verma modules

We start with a graded Lie algebra $\mathcal{L} = \bigoplus_{n \in \mathbb{Z}} \mathcal{L}_n$ such that $\mathcal{L}_0$ is abelian. Denote $\mathcal{L}_+ = \bigoplus_{n>0} \mathcal{L}_n$ and $\mathcal{L}_- = \bigoplus_{n<0} \mathcal{L}_n$.

The main example for us will still be $\widetilde{\mathfrak{g}}$, extended affine Kac-Moody algebra (extension of $\widehat{\mathfrak{g}}$) and Virasoro algebra $Vir \ni d = L_0$.

Now pick $\lambda \in \mathcal{L}_0^*$. To this we can assign the Verma module

$$M(\lambda) = \mathcal{U}(\mathcal{L}) \otimes_{\mathcal{U}(\mathcal{L}_+ + \mathcal{L}_0)} \mathbb{C}_\lambda \underbrace{=}_{\text{as a graded vector space}} \mathcal{U}(\mathcal{L}_-).$$

Then we have the weight space decomposition

$$M(\lambda) = \bigoplus_\nu M(\lambda)_{\lambda - \nu}$$

where $-\nu$ runs over all possible weights (i.e., eigenvalues of $\operatorname{ad} L_0$) of $\mathcal{U}(\mathcal{L}_-)$.

Why are we using $\mathcal{L}_-$ here? The reason is that $\mathcal{L}$ is symmetric and $\mathcal{L}_- \cong \mathcal{L}_+$ are isomorphic. This is not always true, but in this example it is, so in fact we have an antiautomorphism swapping $\mathcal{L}_-$ and $\mathcal{L}_+$,

$$\Theta : \mathcal{U}(\mathcal{L}) \to \mathcal{U}(\mathcal{L})^{\text{op}}, \quad \begin{aligned} &\Theta^2 = \text{id}, \\ &\Theta(\mathcal{L}_n) = \mathcal{L}_{-n}. \end{aligned}$$

**Example 123.**

For Virasoro algebra $Vir$, we have $\Theta(C) = C$ and $\Theta(L_n) = L_{-n}$.

**Example 124.**

For $\tilde{\mathfrak{g}}$, we need to start with a finite-dimensional $\mathfrak{g}$. Here, we want $\Theta$ to swap $e_i$ and $f_i$, and preserve $h_i$. Then we can extend this to $\tilde{\mathfrak{g}}$ by swapping $e_i[r] \leftrightarrow f_i[-r]$, and swapping $h_i[r] \leftrightarrow h_i[-r]$. Lastly we want $c \mapsto c$ and $d \mapsto d$.
In fact the antiautomorphism still has the same form on $\tilde{\mathfrak{g}}$ as on $\mathfrak{g}$ since we could have just defined $f_{\alpha_{\text{highest}}}[1] = e_0 \leftrightarrow f_0 = e_{\alpha_{\text{highest}}}[-1]$.
So we can define this antiautomorphism on any Kac-Moody algebra built from a Cartan matrix.

So in particular it's ok to say $-\nu$ as the weights of $\mathcal{L}_-$. Let $\{\nu\}$ denote all possible weights of $\mathcal{L}_+$. So it's also natural to call the set of all weights in $\mathcal{L}_+$ **roots** and denote them as $\{\alpha\}$. Then any $\nu$ has the form $\nu = \sum_\alpha n_\alpha \alpha$ for $n_\alpha \geq 0$ (note that $\nu$ is a *possible* weight of $\mathcal{L}_+$, not necessarily one that appears).

**Proposition 125.**

Let $v_\lambda \in M(\lambda)$ be a highest weight vector. (This is unique up to scalar, so we have to make some choice.) There is a unique symmetric bilinear form $\langle , \rangle$ on $M(\lambda)$ such that:

1. $\langle v_\lambda, v_\lambda \rangle = 1$.
2. For all $v, v' \in M(\lambda)$ and $Y \in \mathcal{U}(\mathcal{L})$, we have $\langle Yv, v' \rangle = \langle v, \Theta(Y)v' \rangle$.

***Proof.***
First note that $\langle v_\lambda, M(\lambda)_\mu \rangle = 0$ for $\mu \neq \lambda$, as $\Theta$ fixes $\mathcal{L}_0$ and swaps $\mathcal{L}_- \leftrightarrow \mathcal{L}_+$. (More generally, he weight space decomposition is an orthogonal decomposition.) Since $M(\lambda) = \mathcal{U}(\mathcal{L}_-)v_\lambda$ freely, it suffices to define $\langle Y_1 v_\lambda, Y_2 v_\lambda \rangle$ for $Y_1, Y_2 \in \mathcal{U}(\mathcal{L}_-)$. From property (2), it has to be

$$\begin{aligned} \langle Y_1 v_\lambda, Y_2 v_\lambda \rangle &= \langle v_\lambda, \Theta(Y_1) Y_2 v_\lambda \rangle \\ &= \langle v_\lambda, \mathrm{pr}_{M(\lambda)_\lambda} \Theta(Y_1) Y_2 v_\lambda \rangle. \end{aligned}$$

This projection makes sense since the decomposition into weight spaces is an orthogonal decomposition. It remains to see why this is symmetric. But in fact symmetric property follows from uniqueness (else we could just swap the entries and produce a different inner product satisfying the properties), or alternatively from the antiautomorphism, that $\Theta^2 = \mathrm{id}$.

□

Why is this form so cool? Well, it's actually responsible for the irreducibility of the Verma module.

**Proposition 126.**

1. $M(\lambda) = \bigoplus M(\lambda)_{\lambda-\nu}$ is an orthogonal direct sum decomposition.
2. $\mathrm{Rad}\langle,\rangle = N(\lambda) \subset M(\lambda)$, the maximal proper submodule.

***Proof.***
The radical is given by

$$\mathrm{Rad}\langle,\rangle = \left\{v \in M(\lambda) \mid \forall Y \in \mathcal{U}(\mathcal{L}), \quad \mathrm{pr}_{M(\lambda)_\lambda} Yv = 0\right\}.$$

This is a submodule and it is maximal.

□

**Corollary 127.**

$M(\lambda)$ is irreducible $\iff$ on each $M(\lambda)_{\lambda-\nu}$, the restriction of $\langle,\rangle$ is nondegenerate $\iff$ $\mathcal{D}_\nu \neq 0$ for all $\nu$.

**Definition 128.**

The form $\langle,\rangle$ is called the **Shapovalov (contravariant) form**. Let $\Gamma_\nu$ be the matrix of $\langle,\rangle$ in some basis (for example, in the basis $\{Y_i v_\lambda\}$ for a monomial basis $\{Y_i\}$ of $\mathcal{U}(\mathcal{L}_-)$). Let $\mathcal{D}_\nu := \det \Gamma_\nu$.

Now let's say something about these determinants. These guys are some function of $\lambda$, and in fact they are polynomial functions.

**Proposition 129.**

$\mathcal{D}_\nu(\lambda)$ is a polynomial. Moreover, $\deg \mathcal{D}_\nu \leq d_\nu$, where $d_\nu$ are given by some generating functions as follows. Let $\alpha$ be the (positive) roots (=weights of $\mathcal{L}_+$). Then

$$\frac{\mathrm{d}}{\mathrm{d}w} \prod_{\alpha \text{ with multiplicity}} \left(\frac{1}{1-wt^{-1}}\right)\Bigg|_{w=1} = \sum d_\nu t^{-\nu}.$$

***Proof.***
Choose some basis $\{x_\alpha\}$ of $\mathcal{L}_-$. Then (the arrows denote some order)

$$\left\langle \underbrace{\prod_{\rightarrow} x_\alpha^{k_\alpha} v_\lambda}_{Y_1}, \underbrace{\prod_{\rightarrow} x_{-\alpha}^{\ell_\alpha} v_\lambda}_{Y_2} \right\rangle = \left\langle v_\lambda, \prod_{\leftarrow} x_\alpha^{k_\alpha} \prod_{\rightarrow} x_{-\alpha}^{\ell_\alpha} v_\lambda \right\rangle,$$

which is a polynomial in $\lambda$ whose degree is $\leq \min(\deg_{\text{PBW}} Y_1, \deg_{\text{PBW}} Y_2)$. We want to swap all of the factors in the second term from the left to the right, and the contributions will come exactly from the commutators of something in the $\prod_{\leftarrow}$ and something in the $\prod_{\rightarrow}$ So this is a polynomial in $\lambda$, and

$$\deg \mathcal{D}_\nu(\lambda) \leq \sum_{Y \text{ monomials in } \mathcal{U}(\mathcal{L}_-), \text{ weight } -\nu} \deg_{\text{PBW}} Y = d_\nu,$$

where PBW degree is just $\sum k_\alpha$ for $\prod x_\alpha^{k_\alpha}$ (and $\sum \ell_\alpha$ for the other). So to give an upper bound, we just need to sum up over all of the monomials. Therefore

$$\deg \mathcal{D}_\nu(\lambda) \leq \sum_{\substack{\text{monomials } Y \text{ in } \mathcal{U}(\mathcal{L}_-), \\ \text{weight } -\nu}} \deg_{\text{PBW}} Y = d_\nu.$$

This is precisely what is given by the generating function, because the additional variable $w$ counts the degree with respect to the PBW filtration, and we need to sum up all of the degrees with respect to $w$, so we need to differentiate and set $w = 1$. (So $w$ is responsible for PBW.) $\square$

Note: what we mean by the information of the PBW filtration encoded by $w$ can be illustrated here. We can write the character of $S(\mathcal{L}_-)$, which is the same as $\mathcal{U}(\mathcal{L}_-)$, by

$$\text{ch}_{S(\mathcal{L}_-)=\mathcal{U}(\mathcal{L}_-)} = \prod_\alpha \frac{1}{1 - wt^{-\alpha}},$$

since we need to run over all homogeneous generators, but with respect to the PBW degree, each of the generators has degree $1$, so we just have a factor of $w$ above. And after that we want to take a term $w^d t^\nu$ to $d \cdot t^\nu$, hence the differentiation and setting $w = 1$.

**Example 130.**

In $\mathfrak{sl}_2$, we have

$$\frac{\partial}{\partial w} \frac{1}{1 - wt^{-2}}\bigg|_{w=1} = \frac{t^{-2}}{(1 - t^{-2})^2} = t^{-2} + 2t^{-4} + 3t^{-6} + \cdots.$$

So now if we consider the Verma module for $\mathfrak{sl}_2$, represented by a sequence of dots going down, then we have a degree 1 polynomial on the top dot, degree 2 polynomial on the second dot, degree 3 polynomial on the third dot, and so on. In fact we already know this polynomial! That's because we know what values each one vanishes: the polynomials are $\lambda$, $\lambda(\lambda - 1)$, $\lambda(\lambda - 1)(\lambda - 2)$, and so on. So of course these polynomials can have huge degree, but any polynomial down below is divisible by everything upstairs, because if we have a submodule starting on some dot, then the radical has nonzero intersection with that weight space, but it also generates a Verma module starting at that weight space, so there is a nontrivial radical in every dot below it as well. This is a general idea which allows us to compute all of these determinants. For sure it is easy for a finite-dimensional semisimple Lie algebra (and we will do this next time), and not too hard for $\widehat{\mathfrak{sl}_2}$ (and we will also do this next time), but it is a bit more tricky for Virasoro algebra.

# Oct 30

## Shapovalov form on Verma module $M(\lambda)$

**Setup:** Let $\mathcal{L} = \bigoplus_{n\in\mathbb{Z}} \mathcal{L}_n$ be a $\mathbb{Z}$-graded Lie algebra, $\mathcal{L}_0$ is abelian, all $\mathcal{L}_n$ are finite-dimensional, and there is an automorphism of the Lie algebra inducing an anti-automorphism of the universal enveloping algebra:

$$\Theta : \mathcal{U}(\mathcal{L}) \xrightarrow{\sim} \mathcal{U}(\mathcal{L})^{\mathrm{op}}, \quad \Theta(\mathcal{L}_n) = \mathcal{L}_{-n}, \quad \Theta|_{\mathcal{L}_0} = \mathrm{id}.$$

Consider a Verma module $M(\lambda)$ for any highest weight $\lambda \in \mathcal{L}_0^*$. Recall that $M(\lambda) = \mathcal{U}(\mathcal{L}) \otimes_{\mathcal{U}(\mathcal{L}_{\geq 0})} \mathbb{C}_\lambda$. Then there exists a unique (up to constant) symmetric bilinear form

$$\begin{aligned} \langle , \rangle : \quad & M(\lambda) \times M(\lambda) \to \mathbb{C}, \\ & \langle v_\lambda, v_\lambda \rangle \mapsto 1 \quad \text{where } v_\lambda = 1 \otimes e_\lambda \text{ is the highest weight vector,} \\ \forall X \in \mathcal{U}(\mathcal{L}), \quad & \langle Xv, v' \rangle = \langle v, \Theta(X) v' \rangle. \end{aligned}$$

**Corollary 131.**

The weight decomposition $M(\lambda) = \bigoplus_{\lambda'} M(\lambda)_{\lambda'}$ is in fact an orthogonal decomposition.

Now, given $\lambda' = \lambda - \nu$ where $-\nu$ is any weight in the universal enveloping algebra of $\mathcal{U}(\mathcal{L}_-)$, we may consider $\mathcal{D}_\nu(\lambda)$, the determinant of the Gram matrix of $\langle , \rangle$ on $M(\lambda)_{\lambda-\nu}$. We want to consider $\mathcal{D}_\nu(\lambda)$ as a polynomial function of $\lambda$, so that

$$\mathcal{D}_\nu(\lambda) \in \mathbb{C}[\mathcal{L}_0^*] = S(\mathcal{L}_0).$$

**Remark 132.**

Note that $\mathcal{D}_\nu(\lambda)$ is only defined up to a (nonzero) constant, because the Gram matrix itself is only defined up to a nonzero constant. However, we can normalize this matrix by choosing a basis for $\mathcal{L}_-$.

Recall from Corollary 124 that $M(\lambda)$ is irreducible $\iff$ $\mathcal{D}_\nu(\lambda) \neq 0$ for all $\nu$.

What is already known about these determinants? Well, we have an upper bound for the degree, so we know it's a polynomial, where $\deg \mathcal{D}_\nu(\lambda) \leq d_\nu$, where the $d_\nu$ are given by the generating function

$$\frac{\mathrm{d}}{\mathrm{d}w} \prod_{-\alpha:\text{ weights in } \mathcal{L}_-} (1-wt^{-\alpha})\Big|_{w=1} = \sum d_\nu t^{-\nu}.$$

Technically, we haven't even confirmed that the $\mathcal{D}_\nu(\lambda)$ are nonzero. However, under mild assumptions, we can confirm that these are indeed not all zero.

**Let us assume** that $[,]: \mathcal{L}_n \times \mathcal{L}_{-n} \to \mathcal{L}_0$ is nondegenerate.
We can regard this as a pairing $\mathcal{L}_n \times \mathcal{L}_{-n}$, with values in $\mathbb{C}[\mathcal{L}_0^*] \supset \mathcal{L}_0$ (we regard $\mathcal{L}_0$ as the subset of linear functions inside $\mathbb{C}[\mathcal{L}_0^*]$; this will be useful to work with a ring rather than a vector space).

**Example 133.**

This is indeed the case for pretty much all examples we already know: $\widehat{\mathfrak{g}}$, $Vir$, and $\mathfrak{a}$ (Heisenberg).

Since this is pretty much always the case in the algebras we consider, we can assume this.

**Proposition 134.**

Under the assumption, then $\mathcal{D}_\nu(\lambda) \neq 0$, and moreover $\deg \mathcal{D}_\nu(\lambda) = d_\nu$.

***Proof.***
What we'll do is a version of the deformation argument for the (ir)reducibility of the Verma module. Consider the family of Lie algebras $\mathcal{L}^\varepsilon$, depending on $\varepsilon$. As a vector space, $\mathcal{L}^\varepsilon \cong \mathcal{L}$, but the commutator operation gets renormalized. In fact, what happens is we take some basis in $\mathcal{L}$ and renormalize it, by multiplying by powers of $\varepsilon$. We have:

$$\begin{gathered}
\mathcal{L}_+ \ni x \mapsto x^\varepsilon := \varepsilon \cdot x \in \mathcal{L}_+^\varepsilon, \\
\mathcal{L}_0 \ni x \mapsto x^\varepsilon := \varepsilon^2 \cdot x \in \mathcal{L}_0^\varepsilon, \\
\mathcal{L}_- \ni x \mapsto x^\varepsilon := \varepsilon \cdot x \in \mathcal{L}_-^\varepsilon.
\end{gathered}$$

For $x_n \in \mathcal{L}_n$ and $x_{-n} \in \mathcal{L}_{-n}$, we have $[x_n^\varepsilon, x_{-n}^\varepsilon] = [x_n, x_{-n}]^\varepsilon$, and for $n+m \neq 0$ we have $[x_n^\varepsilon, x_m^\varepsilon] = \varepsilon \cdot [x_n, x_m]^\varepsilon$.

This is a family of Lie algebras so that $\mathcal{L}^\varepsilon \cong \mathcal{L}$ for $\varepsilon \neq 0$, but for $\varepsilon = 0$ we get something like the Heisenberg algebra, where all of the commutators are zero except when $n+m=0$.

Now, we can still write the determinant of the Shapovalov form $\mathcal{D}_\nu^\varepsilon(\lambda)$ for $\mathcal{L}^\varepsilon$.

**Lemma 135.**

$$\mathcal{D}_\nu^\varepsilon(\lambda) = \mathcal{D}_\nu(\varepsilon^{-2}\lambda) \cdot \varepsilon^{2d_\nu}.$$

The proof is a straightforward computation.

**Corollary 136.**

$\mathcal{D}_\nu^0(\lambda)$ is the degree-$d_\nu$ term of $\mathcal{D}_\nu(\lambda)$.

However, for $\varepsilon = 0$, the Shapovalov form is really easy to compute in the monomial basis.

**Lemma 137.**

$$\mathcal{D}_\nu^0(\lambda) = \prod_{\nu = \sum \underbrace{k_\alpha}_{\in \mathbb{Z}_{\geq 0}} \cdot \underbrace{\alpha}_{\text{roots}}} \prod_\alpha \det S^{k_\alpha} \omega_\alpha, \quad \omega_\alpha : L_\alpha \times L_{-\alpha} \to L_0.$$

***Proof.***
Let $Y_1 = \prod_\alpha \prod_{i=1}^{k_\alpha^1} x_{\alpha,i}^1$ where $x_{\alpha,i} \in \mathcal{L}_{-\alpha}$, and $Y_2 = \prod_\alpha \prod_{j=1}^{k_\alpha^2} x_{\alpha,j}^2$ the same thing; these are two monomials. Then

$$\langle Y_1 v \lambda, Y_2 v_\lambda \rangle = \begin{cases} 0 & \exists\, \alpha \text{ such that } k_\alpha^1 \neq k_\alpha^2, \\ \bigotimes S^{k_\alpha} \omega_\alpha & \text{otherwise.} \end{cases}$$

In the first case, we just apply $\Theta$, and get $0$ because $[x_n^0, x_m^0] = 0$ for $n + m \neq 0$. This proves the lemma.

□

The proposition follows from the lemma.

□

**Corollary 138.**

$M(\lambda)$ is irreducible for Weil-generic $\lambda$.

Now, so far this is mildly disappointing: we know that the $\mathcal{D}_\nu(\lambda)$ are huge polynomials, but we know almost nothing else about them. So we'd really like to compute $\mathcal{D}_\nu(\lambda)$.

**Proposition 139.**

Suppose $F$ is an irreducible factor of $\mathcal{D}_\nu(\lambda)$ (regarded as a multivariate polynomial). Suppose $F$ has multiplicity $m$, i.e. $F^m|\mathcal{D}_\nu(\lambda)$ but $F^{m+1}\nmid\mathcal{D}_\nu(\lambda)$. Then for all $\nu'$ such that $\nu-\nu'$ is a weight of $\mathcal{U}(\mathcal{L}_-)$, we have

$$F^{m\cdot p_{\nu'-\nu}}|\mathcal{D}_{\nu'}(\lambda), \quad p_{\nu'-\nu}=\dim\mathcal{U}(\mathcal{L}_-)_{\nu-\nu'}.$$

**Remark 140.**

This divisibility is not necessarily strict. In other words, there may be a higher power of $F$ dividing $\mathcal{D}_{\nu'}(\lambda)$.

We'll prove this next time. The idea of the proof: consider the Verma module as a downwards cone with $v_\lambda$ at the top. Then once we have a weight space $\lambda-\nu$ where the Shapovalov form degenerates, then there is a submodule in $M(\lambda)$ which intersects with this weight space nontrivially. But then for any further weight space $\lambda-\nu'$ in the weight spaces of this submodule, this submodule intersects with this weight space as well.

# Nov 1

## Determinant of Shapovalov form

First, let's correct a statement from last time which is wrong.

Let $\mathcal{L}$ be our graded Lie algebra with Shapovalov form $\langle -,-\rangle$.

**Definition 141.**

Define $\mathcal{D}_\nu(\lambda):=\det\langle,\rangle|_{M(\lambda)_{\lambda-\nu}}$. Note that $\mathcal{D}_\nu(\lambda)\in\mathbb{C}[\mathcal{L}_0^*]=:R$, which is a factorial ring.

**Proposition 142.**

Suppose $F$ is a prime factor of $\mathcal{D}_\nu(\lambda)$. Then for any other $\nu'$, we have $F^{p_{\nu-\nu'}}|\mathcal{D}_{\nu'}(\lambda)$, where $p_{\nu-\nu'}=\dim\mathcal{U}(\mathcal{L}_-)_{\nu-\nu'}$.

Note that the statement from last lecture is not true: the above statement rectifies it.

***Proof.***

The strategy of the proof is as follows. Consider the module $M(\lambda)$. Then the divisibility by an irreducible polynomial means that for any $\lambda$ such that $F(\lambda)=0$, then we have a submodule which intersects nontrivially with the $\nu$-weightspace, i.e. weight $\lambda-\nu$. Take any vector from this intersection and generate a free $\mathcal{U}(\mathcal{L}_-)$-submodule. Then the dimension of its intersection with the weight space $\lambda-\nu'$ is $p_{\nu'-\nu}$ because it is a free $\mathcal{U}(\mathcal{L}_-)$-submodule. (We used that a submodule of a free $\mathcal{U}(\mathcal{L}_-)$-module is again free; this is a consequence of PBW theorem.) So at least intuitively, $\mathcal{D}_{\nu'}(\lambda)$ should be divisible by $F^{p_{\nu'-\nu}}$, because we have $p_{\nu'-\nu}$ many polynomials in that weight space, each "giving" an $F$.

**Want**: consider $M(\lambda)$ universally, over $R$, where $\langle,\rangle$ is an $R$-value bilinear form. Pick a basis $\{u_1,\dots,u_N\}$ over $M(\lambda)_{\lambda-\nu}$ such that $F|\langle u_1,-\rangle$.
Once we have such a basis, we can produce a basis in $M(\lambda)_{\lambda-\nu'}$ by doing the following: we take $u_1$ and apply all possible monomials $Y_i\in\mathcal{U}(\mathcal{L}_-)_{\nu-\nu'}$ to get $\{Y_iu_1\}$, then complete it to a basis of $M(\lambda)_{\lambda-\nu'}$. Then the determinant of the Shapovalov form in such a basis will be divisible by that power of $F$. (This is because the construction of $u_i$ means that for all $Y\in\mathcal{U}(\mathcal{L})$, then $F|\mathrm{pr}_{v_\lambda}Yu_i$. Therefore the same is true of the $Y_iu_1$.)

**Problem**: we are dealing with free $R$-modules, not vector spaces over a field.
**Idea**: reduce this to dealing with a PID, rather than just any factorial ring.
First, we want to use the Shapovalov form $\langle,\rangle$ to get a map $\mathrm{Sh}:M(\lambda)\to M(\lambda)^\vee$, the restricted dual. Note that $M(\lambda)^\vee$ has a $\mathcal{U}(\mathcal{L})$-module structure given by $(X.\,\xi)(v)=\xi(\Theta(X).\,v)$ for $\xi\in M(\lambda)^\vee$ and $X\in\mathcal{U}(\mathcal{L})$. This is called the **contragradient dual module**. In fact, in general the contragredient dual module is not even in the category $\mathcal{O}$, as it can be infinitely generated; this is because modules can be not Artinian. In this case it is indeed in category $\mathcal{O}$.
Then for any weight space we get

$$\mathrm{Sh}(M(\lambda)_{\lambda-\nu})\subset M(\lambda)^\vee_{\lambda-\nu}.$$

For generic $\lambda,\nu$ this is an isomorphism; in general we want to find the codimension.

Second, consider the universal Verma module over $R$:

$$M=\mathcal{U}(\mathcal{L})\otimes R\otimes_{\mathcal{U}(\mathcal{L}_{\geq 0})\otimes R}R.$$

This is still a free $\mathcal{U}(\mathcal{L}_-)\otimes R$-module, and in particular we still get a weight decomposition $M=\bigoplus M_{-\nu}$. (Here, the $\otimes_{\mathcal{U}(\mathcal{L}_{\geq 0})\otimes R}R$ means that $\mathcal{L}_+$ acts by $0$, $h\in\mathcal{L}_0$ acts by $h\in R=S(\mathcal{L}_0)$.)

Now consider $R_{loc}=R_{(F)}$ localization, along with $M_{loc}L=M\otimes_R R_{loc}$ and $M^\vee_{loc}:=M^\vee\otimes_R R_{loc}$. Then $\mathcal{D}^{loc}_\nu\in R_{loc}$. We have $F^m|\mathcal{D}_\nu \text{ in } R \iff F^m\mathcal{D}^{loc}_v \text{ in } R_{loc}$, because $R$ is factorial. This means we can just look at divisibility in localizations.

Third, we can pick a compatible basis of $\mathrm{Sh}(M_{loc,\nu})$ and $M^\vee_{loc,\nu}\ni v_1,\dots,v_N$ such that $F^{m_1}v_1,\dots,F^{m_N}v_N$ is a basis of $\mathrm{Sh}(M_{loc,-\nu})$ where $F^{m_i}v_i=\mathrm{Sh}(u_i)$ for some $u_i\in M_{loc,-\nu}$ . We can do this because $R_{loc}$ is a PID. And all of this is equivalent to the condition that $F^{\sum m_i}|\mathcal{D}^{loc}_\nu$.

Finally, we can assume $m_1 > 0$. Then $M_{loc,-\nu'}$ has a basis of the form $w_1 = Y_1 u_1, \dots, w_p = Y_p u_1, w_{p+1}, \dots, w_q$ where $Y_1, \dots, Y_p$ is a basis of $\mathcal{U}(\mathcal{L}_-)_{\nu-\nu'}$.

Then $\mathrm{Sh}(w_1), \dots, \mathrm{Sh}(w_p) \in F \cdot M^\vee_{loc,-\nu'}$, which implies that $F^p | \mathcal{D}^{loc}_{-\nu'}$.

□

How does this result help us to decompose the determinant of the Shapovalov form? Well, $\mathcal{D}_\nu(\lambda) = F_\nu F_{?\nu}$ where $F_\nu$ are the new factors, and $F_{?\nu}$ is the product of the factors that occur in previous weight spaces.

### Upper bound for $\deg \mathcal{D}_\nu$

We already have an upper bound for $\deg \mathcal{D}_\nu$, and it's given by the power series

$$\frac{\partial}{\partial w}\bigg|_{w=1} \prod \frac{1}{1 - wt^{-\alpha}}.$$

On the other hand, each of the irreducible factors gives rise to the same irreducible factors with multiplicities being the coefficients of the power series $\prod \frac{1}{1-t^{-\alpha}} = \sum p_\nu t^{-\nu}$. So to have this upper bound, we just need to divide:

$$\frac{\frac{\partial}{\partial w}\Big|_{w=1} \prod \frac{1}{1-wt^{-\alpha}}}{\prod \frac{1}{1-t^{-\alpha}} = \left(\prod \frac{1}{1-wt^{-\alpha}}\Big|_{w=1}\right)} = \frac{\partial}{\partial w}\bigg|_{w=1} \log \prod \frac{1}{1 - wt^{-\alpha}} = \sum_\alpha \frac{t^{-\alpha}}{1 - t^{-\alpha}}.$$

Then we will see next time that for finite-dimensional semisimple Lie algebra $\mathfrak{g}$, this is already sufficient for decomposing the determinant into linear factors, because all of the roots have multiplicity $1$, and all the factors coming from each nonzero coefficient of this power series is different, so this is not just an upper bound, but an equality.

# Nov 3

Today we will continue with irreducibility of Verma modules depending on its highest weight.

## Irreducibility criterion for $M(\lambda)$

Let's summarize what we already know. We know that $M(\lambda)$ is irreducible $\iff \mathcal{D}_\nu(\lambda) \neq 0$ for all $\nu$. We also know that

$$\mathcal{D}_\nu(\lambda) = \prod_{\substack{\nu' = m\alpha \\ \alpha \text{ root}}} F_{\nu'}^{p_{\nu-\nu'}}.$$

These new factors only come up with $\nu'$ is a multiple of a root $\alpha$. This is because last time, we wrote a generating function for the degrees of new factors in the Shapovalov determinant, which was

$$\frac{\mathrm{d}}{\mathrm{d}w}\bigg|_{w=1} \log \prod_{\alpha \text{ positive roots}} \frac{1}{1-wt^{-\alpha}} = \sum_{\alpha} \frac{t^{-\alpha}}{1-t^{-\alpha}} = \sum_{m\geq 1,\, \alpha} t^{-m\alpha}.$$

This means that $\lambda - m\alpha$ are the only possible weights where we can expect new factors in the Shapovalov form.

## Examples

Let's see some examples: $\mathfrak{sl}_2$, $\mathfrak{sl}_3$. Let's start with $\mathfrak{sl}_2$, which is the easiest example ever.

**Example 143.**

For $\mathfrak{g} = \mathfrak{sl}_2$, we have the following explicit description of $M(\lambda)$. There is only one root $\alpha = 2$. Then each weight space $\lambda, \lambda-2, \lambda-4, \ldots$ is one dimensional. The Shapovalov determinant at weight $\lambda$ is 1. The determinant at weight $\lambda - 2$ is $\lambda$. The determinant at weight $\lambda - 4$ is $\lambda(\lambda-1)$. The determinant at $\lambda - 6$ is $\lambda(\lambda-1)(\lambda-2)$. And so on.

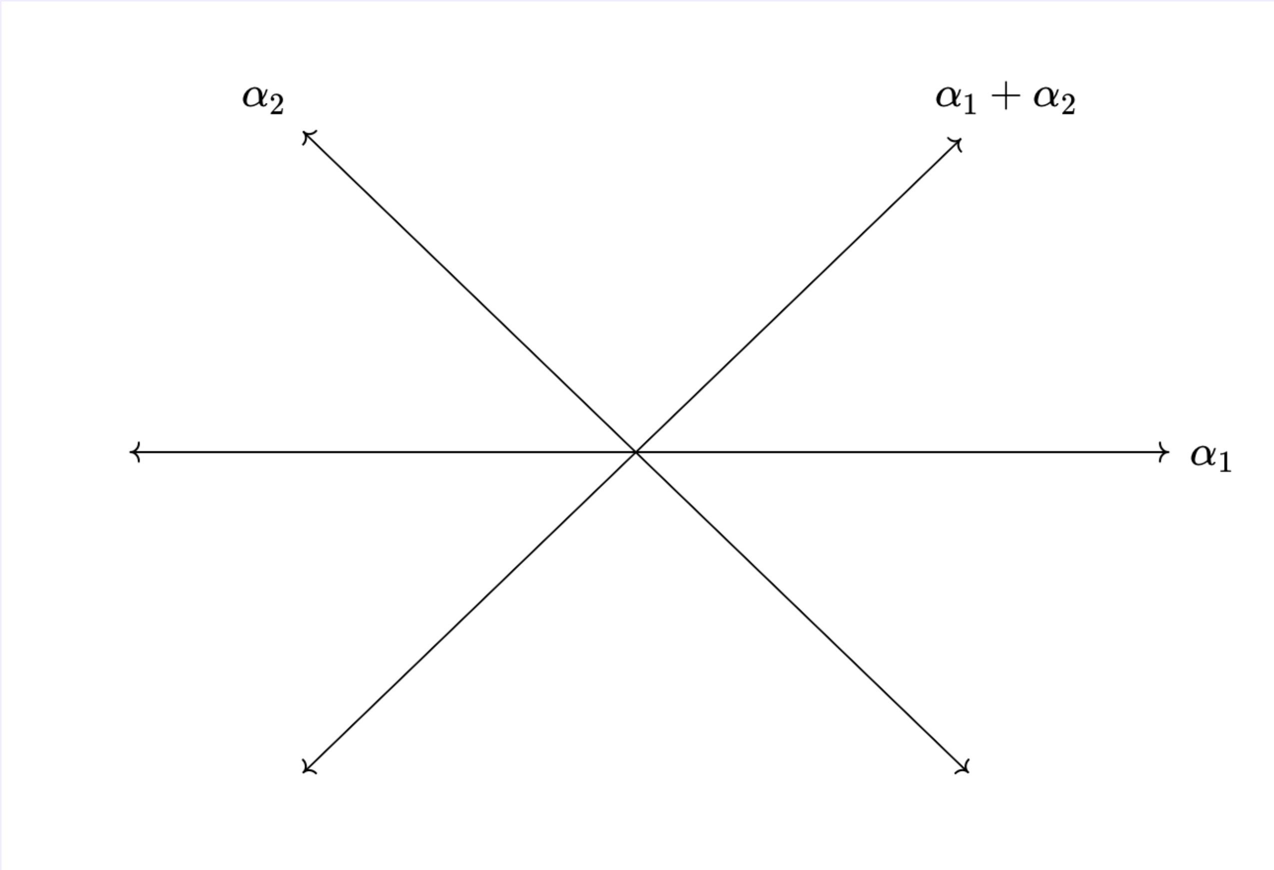
The reason that these factors arise is actually quite simple. These weight spaces are one-dimensional, so it's already straightforward, but in fact we don't even need to compute these matrix elements explicitly. The reason is that for $\lambda = 0$, the weight space $-2$, i.e. $M(0)_{-2}$ generates a submodule isomorphic to $M(-2) \subset M(0)$. This is because $M(0)$ has a one-dimensional quotient, the simple module $L(0)$. Therefore, going down one level from weight $\lambda$ to $\lambda - 2$ must introduce a factor in the Shapovalov determinant which vanishes at 0, namely $\lambda$: therefore the Shapovalov determinant of weight $\lambda - 2$ must have a factor of $\lambda$. On the other hand, we know that the degree of the Shapovalov determinant is 1, so it must be exactly $\lambda$ (up to nonzero scalar).

Similarly, if $\lambda = 1$, then we have a submodule $M(-3) \subset M(1)$ such that the quotient is $M(1)/M(-3) = L(1)$, a two-dimensional representation with highest weight 1. This means that moving from weight $\lambda - 2$ to $\lambda - 4$, we introduce a factor in the Shapovalov determinant which is killed by 1, namely $\lambda - 1$. On the other hand, the degree of the Shapovalov determinant is at most 2 . Therefore the Shapovalov determinant is exactly $\lambda(\lambda-1)$. Continuing in this way, we can compute the Shapovalov determinants for every weight space $\lambda - 2m$.

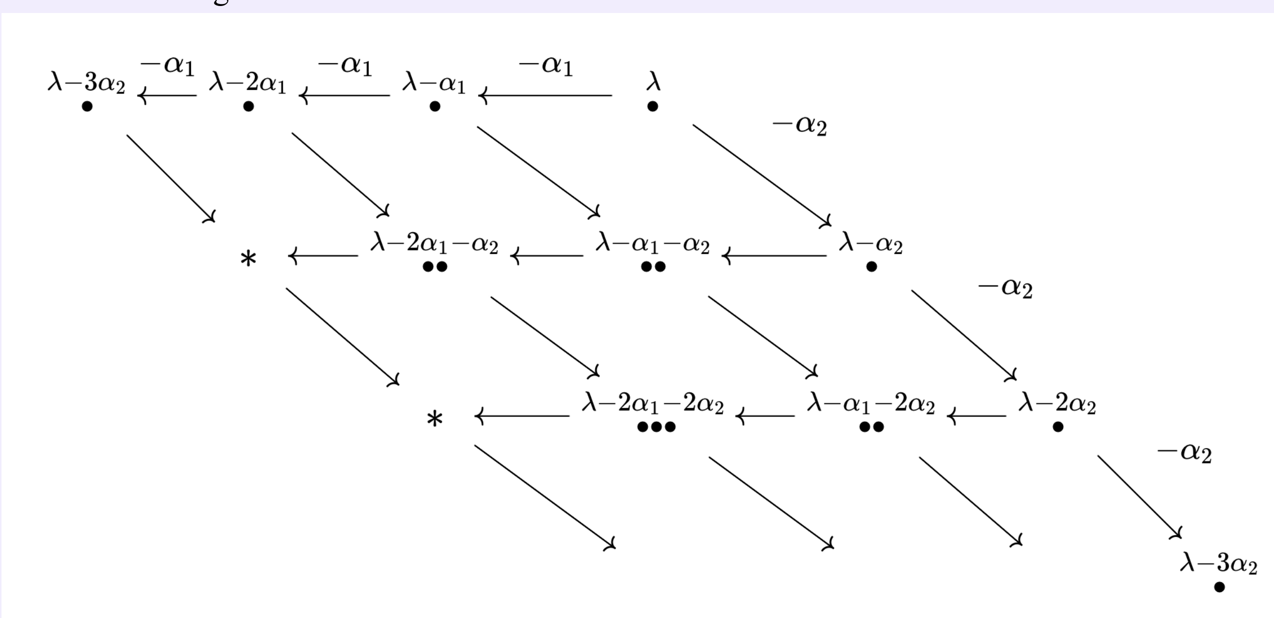
Now let's look at what happens for $\mathfrak{sl}_3$.

**Example 144.**

Let $\mathfrak{g} = \mathfrak{sl}_3$. Recall that the root system looks like:




Therefore the negative roots act like:




The dimensions of the weight spaces are indicated by the number of bullets, and they're precisely the number of ways to write that weight as nonnegative integer combinations of the positive roots $\alpha_1, \alpha_2, \alpha_3 = \alpha_1 + \alpha_2$. (For example, the weight space $\lambda - \alpha_1 - \alpha_2$ has dimension 2, by coefficients $(1, 1, 0)$ and $(0, 0, 1)$, which are linearly independent by PBW theorem.) See the

[Kostant partition function](#); the dimension of a weight space $\mu$ can be computed by writing $mu = \lambda - n_1\alpha_1 - n_2\alpha_2$ for $n_1, n_2 \in \mathbb{Z}_{\geq 0}$, then $\dim M(\lambda)_\mu = 1 + \min(n_1, n_2)$.

The weight spaces which introduce a new factor in the Shapovalov determinant are precisely the starred weight spaces: they are $\lambda - m \cdot \alpha_i$ for one of the positive roots $\alpha_i \in \{\alpha_1, \alpha_2, \alpha_3\}$.

$$
\begin{tikzcd}
\underset{\star}{\lambda-3\alpha_2} \arrow[dr] & \underset{\star}{\lambda-2\alpha_1} \arrow[l, "-\alpha_1"'] \arrow[dr] & \underset{\star}{\lambda-\alpha_1} \arrow[l, "-\alpha_1"'] \arrow[dr] & \underset{\star}{\lambda} \arrow[l, "-\alpha_1"'] \arrow[dr, "-\alpha_2"] & & & \\
& {} \arrow[dr] & {} \arrow[l] \arrow[dr] & \underset{\star}{\lambda-\alpha_1-\alpha_2} \arrow[l] \arrow[dr] & \underset{\star}{\lambda-\alpha_2} \arrow[l] \arrow[dr, "-\alpha_2"] & & \\
& & {} \arrow[dr] & \underset{\star}{\lambda-2\alpha_1-2\alpha_2} \arrow[l] \arrow[dr] & {} \arrow[l] \arrow[dr] & \underset{\star}{\lambda-2\alpha_2} \arrow[l] \arrow[dr, "-\alpha_2"] & \\
& & & \underset{\star}{\lambda-3\alpha_1-3\alpha_2} & {} \arrow[l] & {} \arrow[l] & \underset{\star}{\lambda-3\alpha_2} \arrow[l]
\end{tikzcd}
$$

Why do these come up, and what factors do they introduce? This question essentially boils down to where singular vectors may exist - wherever they may come up, they impose a condition on $\lambda$ which, if satisfied, creates a singular vector, and the corresponding factor in the Shapovalov determinant reflects exactly this condition.

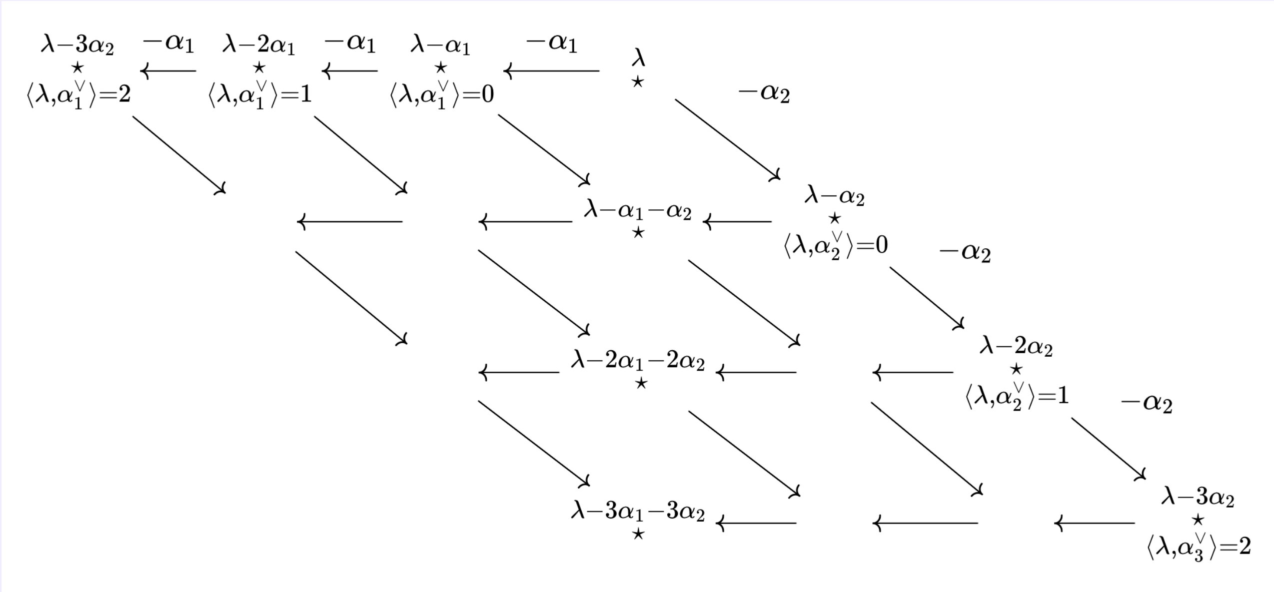
As a case study, let's look at $v = f_{\alpha_1}^2 v_\lambda$, the generator of the weight space $\lambda - 2\alpha_1$. We already know that $e_{\alpha_2} v = 0$. In order for $v$ to be singular, we also need $e_{\alpha_1} v = 0$. But this implies that

$$0 = e_{\alpha_1} v = e_{\alpha_1} f_{\alpha_1}^2 v_\lambda,$$

which is a question purely about the $\mathfrak{sl}_2$-triple generated by root $\alpha_1$ acting on the highest vector $v_\lambda$. So this vector is 0 iff the weight $\lambda$ restricted to the copy of $\mathfrak{sl}_2$ generated by this root $\alpha_1$ is 1:

$$0 = e_{\alpha_1} f_{\alpha_1}^2 v_\lambda \iff 1 = \lambda\Big|_{\mathfrak{sl}_2^{\alpha_1}} = \langle \lambda, h_{\alpha_1} \rangle = \frac{2\langle \lambda, \alpha_1 \rangle}{\langle \alpha_1, \alpha_1 \rangle} = \langle \lambda, \alpha_1^\vee \rangle.$$

So the condition that there is a singular vector at weight $\lambda - 2\alpha_1$ is precisely that $\langle \lambda, \alpha_1^\vee \rangle = 1$. More generally, $f_{\alpha_i}^k v_\lambda$ is a singular vector $\iff \langle \lambda, \alpha_i^\vee \rangle - k + 1 = 0$. This can be rewritten as $\langle \lambda + \rho, \alpha_i^\vee \rangle = k$, where $\rho \in \mathfrak{h}^*$ such that $\langle \rho, \alpha_i^\vee \rangle = 1$ for all $i$, and $k \in \mathbb{Z}_{\geq 0}$; this will be useful when we need to consider the dot action of the Weyl group $W$. This is summarized in this diagram, with the condition written under the star:




Now we see that these are polynomial conditions on $\lambda$, and furthermore they are all distinct, hence the factors introduced in the Shapovalov determinant are all distinct, and thus covers every factor introduced by simple roots $\alpha_1, \alpha_2$. There is one more root: $\alpha_3 = \alpha_1 + \alpha_2$, and this is a bit trickier. This is because when you apply $f_{\alpha_3}$ to $v_\lambda$ repeatedly, you actually *never* get a singular vector. But for some reason, under some conditions on $\lambda$, you *do* get a singular vector in these weight spaces. The idea is to examine submodules in $M(\lambda)$ for integral $\lambda$.

Let's first consider the dominant case. Let $\lambda$ be dominant: $\langle \lambda + \rho, \alpha_i^\vee \rangle =: m_i > 0$, where $m_i \in \mathbb{Z}_{>0}$ . We just showed that we have submodules generated by the vectors at weight spaces $\lambda - m_1\alpha_1$ and $\lambda - m_2\alpha_2$. It turns out that these two Verma modules intersect nontrivially, for dimension reasons: each is a free $\mathcal{U}(\mathfrak{n}_-)$ module, just shifted by the corresponding weight. If we check the dimensions of the weight subspaces, the Kostant partition function shows that $\dim \mathcal{U}(\mathfrak{n}_-)_{-n_1\alpha_1 - n_2\alpha_2} = 1 + \min(n_1, n_2)$, and eventually the sum of the weight spaces of these two submodules must be larger than the corresponding weight space of the original $M(\lambda)$. (You can think of the dimensions of the weight spaces as growing linearly with some fixed coefficient; then summing the dimensions of the two submodules doubles the coefficient at the cost of some fixed finite offset, and for large enough terms, $2x + c$ overtakes $x + c'$ even when $c' \gg c$.) On the other hand, there are not that many possibilities for where these can intersect (at least at the highest weight): they must generate a submodule in the original Verma module, so where they intersect should be special. This naturally leads us to our next question.

What is the highest weight of a submodule in $M(\lambda)$?
The answer is known: it's $\mu$ such that (where $Z$ denotes center) $Z(\mathcal{U}(\mathfrak{sl}_3)|_{M(\mu)} = Z(\mathcal{U}(\mathfrak{sl}_3))|_{M(\lambda)}$. That's because the center acts by a scalar in $M(\lambda)$, so must act by the same scalar on any submodule, so if $M(\mu)$ is a submodule then the scalar must still act as the same scalar. (Note that in the $\mathfrak{sl}_2$ case, this already answers the question, since the center is generated by the Casimir, which acts by $\frac{\lambda(\lambda+2)}{2}$ on $M(\lambda)$. But in this case, it is slightly more complicated!)

Now, there is a very general description of the center of the universal enveloping algebra: the Harish-Chandra theorem states that $Z(\mathcal{U}(\mathfrak{g})) = \mathcal{U}(\mathfrak{g})^{\mathfrak{g}} \xleftarrow{\sim} S(\mathfrak{g})^{\mathfrak{g}}$, so $\operatorname{gr} Z(\mathcal{U}(\mathfrak{g})) = S(\mathfrak{g})^{\mathfrak{g}}$. In $\mathfrak{g} = \mathfrak{sl}_3$ case, it's generated by two elements (in degrees 2 and 3): $\operatorname{tr} X^2$ and $\operatorname{tr} X^3$ in $\mathbb{C}[X \in \operatorname{Mat}_3(\mathbb{C}) \mid \operatorname{tr} X = 0]^{\mathfrak{sl}_3}$. Now Harish-Chandra says that $Z(\mathcal{U}(\mathfrak{g}))_M = \mathbb{C}[\mathfrak{h}^*]^{(W,\cdot)}$, where the dot action is $w \cdot \lambda = w(\lambda + \rho) - \rho$.

In particular, for all $c \in Z(\mathcal{U}(\mathfrak{g}))$, then $c$ acts by a scalar, and this scalar $c|_{M(\lambda)}$ is a $(W, \cdot)$-invariant polynomial of $\lambda$. In our case of $\mathfrak{sl}_3$, they are generated by $x_1^2 + x_2^2 + x_3^2$ and $x_1^3 + x_2^3 + x_3^3$, but shifted by $\rho$. So if $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ then $\lambda + \rho = (\lambda_1 + \rho_1, \lambda_2 + \rho_2, \lambda_3 + \rho_3)$, and the $(W, \cdot)$ action sends $\lambda \mapsto (w(\lambda_1 + \rho_1) - \rho_1, w(\lambda_2 + \rho_2) - \rho_2, w(\lambda_3 + \rho_3) - \rho_3)$, and the polynomials are precisely the polynomials in terms of these new factors.

So returning to our picture, the only possible highest weights are $W \cdot \lambda$. In our case $W = S_3$ so there are six possible $\mu$, and they correspond to permutations, i.e. permutations of $(\lambda_1 + \rho_1, \lambda_2 + \rho_2, \lambda_3 + \rho_3)$, then subtracting $(\rho_1, \rho_2, \rho_3)$ (just work out the dot action of $S_3$).

It is even possible to show that an intersection of two submodules is a sum of the Verma modules of these highest weights.

What we find is that $\lambda - m_1\alpha_1$ and $\lambda - m_2\alpha_2$ generate submodules, and these intersect to form further submodules: the highest weights of these will differ from $\lambda - m_1\alpha_1$ and $\lambda - m_2\alpha_2$ by some positive integer multiples of the highest weight $\alpha_3 = \alpha_1 + \alpha_2$, arranged by the $W = S_3$ dot action. (The difference could be any multiple of the highest root.) There are infinitely many integral $\lambda$ such that $\langle \lambda + \rho, (\alpha_1 + \alpha_2)^\vee \rangle = k \in \mathbb{Z}_{>0}$.

So the condition is: fix $\alpha$ and $k \in \mathbb{Z}_{>0}$. There is a singular vector in $M(\lambda)_{\lambda - k\alpha} \iff \langle \lambda + \rho, \alpha^\vee \rangle - k = 0$.

**Theorem 145** (Verma).

Let $\mathfrak{g} = \mathfrak{sl}_3$. Then $M(\lambda)$ is irreducible $\iff \langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{>0}$ for all positive roots $\alpha$.

We will generalize this theorem next time, both to finite-dimensional cases, and discussing for the infinite-dimensional case (as well as the special case of $\widehat{\mathfrak{sl}_2}$).

# Nov 6

Vasily Krylov will be teaching this class instead of Leonid Rybnikov.

**Notation:**

- $\mathfrak{g}$ is a simple finite-dimensional Lie algebra.
- $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra.
- $\Phi^+$ denotes the positive roots.
- $\Lambda$ denotes the weight lattice (this consists of $\lambda \in \mathfrak{h}^*$ such that $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$ for all $\alpha \in \Phi$).
- $\Delta \subset \Phi^+$ denotes the simple roots.
- $\leq$ is the partial order on $\Lambda$. Weights $\mu \leq \lambda \iff \lambda - \mu \in \mathbb{Z}_{\geq 0}\Delta$.
- $W$ denotes the Weyl group.

Our goal for today is to prove the following theorem:

**Theorem 146** (Verma)**.**

If $\lambda \in \mathfrak{h}^*$ and $\alpha \in \Phi^+$ are such that $s_\alpha \cdot \lambda =: \mu \leq \lambda$, then there is an embedding $M(\mu) \subset M(\lambda)$.

What we already know:

**Proposition 147.**

In the assumptions of Theorem 143 (Verma), if $\alpha$ is simple (i.e. $\alpha \in \Delta$), then there exists $M(\mu) \subset M(\lambda)$.

**Corollary 148.**

If $\lambda + \rho \in \Lambda^+$ is dominant, then for all $w \in W$, then $M(w \cdot \lambda) \subset M(\lambda)$.
In fact, writing $w = s_n s_{n-1} \cdots s_1$ as a reduced expression, then

$$\begin{gathered} w \cdot \lambda \leq (s_{n-1} \cdots s_1) \cdot \lambda \leq \cdots \leq s_1 \cdot \lambda \leq \lambda \\ \Downarrow \\ M(w \cdot \lambda) \subset M((s_{n-1} \cdots s_1) \cdot \lambda) \subset \cdots \subset M(s_1 \cdot \lambda) \subset M(\lambda). \end{gathered}$$

**Naive idea:** prove Theorem 143 (Verma) step by step, by writing $s_\alpha = s_n \cdots s_1$ as a product of simple reflections.

Unfortunately, this doesn't work. Let's take a look at a problem which arises. Let $\mathfrak{g} = \mathfrak{sl}_3$. Then $\Delta = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3\}$, and $\rho = \alpha_1 + \alpha_2$. Pick $\alpha = \alpha_1 + \alpha_2 = \varepsilon_1 - \varepsilon_3$. Now let us write:

$$\begin{aligned} \lambda &:= s_{\alpha_1} \cdot 0 = s_{\alpha_1}(\rho) - \rho = -\langle \rho, \alpha_1^\vee \rangle \alpha_1 = -\alpha_1, \\ \mu &:= s_\alpha \cdot \lambda = s_\alpha(\alpha_2) - \rho = -\alpha_1 - \langle \alpha_2, \alpha^\vee \rangle \alpha = -2\alpha_1 - \alpha_2. \end{aligned}$$

We easily see that $\lambda - \mu = \alpha_1 + \alpha_2 = \alpha \in \Delta \subset \mathbb{Z}_{\geq 0}\Delta$, so $\mu \leq \lambda$. By Theorem 143 (Verma), we should have an embedding $M(\mu) \subset M(\lambda)$. However, if we apply our naive idea to try to prove this, we should first write $s_\alpha$ as a product of simple reflections:

$$s_\alpha = s_{\alpha_1} s_{\alpha_2} s_{\alpha_1}.$$

But we **do not** get a chain as in Corollary 145: we have

$$\underbrace{s_\alpha \cdot \lambda}_{-2\alpha_1 - \alpha_2} < \underbrace{s_{\alpha_2} s_{\alpha_1} \cdot \lambda}_{-\alpha_2} < \underbrace{s_{\alpha_1} \cdot \lambda}_{0} > \underbrace{\lambda}_{-\alpha_1}.$$

So we see that the dominant hypothesis is actually crucial, and also that we unfortunately won't be able to get this naive idea to work. Let's use a different approach. (This approach will be more tricky.)

**Lemma 149.**

Let $\mathfrak{y}$ be a nilpotent Lie algebra, $x \in \mathfrak{y}$, $u \in \mathcal{U}(\mathfrak{y})$. Given $n \in \mathbb{Z}_{\geq 0}$, then there exists $t \in \mathbb{Z}_{\geq 0}$ such that $x^t u \in \mathcal{U}(\mathfrak{y}) x^n$.

***Proof.***

First, write the operator $\operatorname{ad} x : \mathcal{U}(\mathfrak{y}) \to \mathcal{U}(\mathfrak{y})$ as $\operatorname{ad} x = \ell_x - r_x$, the difference of two commuting operators given by left and right multiplication by $x$. It immediately follows that $r_x$ commutes with $\operatorname{ad} x$. Then since $\mathfrak{y}$ is nilpotent, there exists $q > 0$ such that $(\operatorname{ad} x)^q u = 0$ (in fact, there even exists $q$ such that $(\operatorname{ad} x)^q = 0$ as operators, but this is enough for our purposes). Now let's choose $t \geq q + n$. Then

$$\begin{aligned} x^t u &= \ell_x^t u, \\ &= (r_x + \operatorname{ad} x)^t u, \\ &= \sum_{i=0}^{t} \binom{t}{i} r_x^{t-i} (\operatorname{ad} x)^i u, \\ &= \sum_{i=0}^{q} \binom{t}{i} (\operatorname{ad} x)^i u x^{t-i} \in \mathcal{U}(\mathfrak{y}) x^{t-q} \subset \mathcal{U}(\mathfrak{y}) x^n. \end{aligned}$$

□

We'll use this lemma later. Now let us formulate another fact; this one will be left as an exercise (use induction on $t$).

**Lemma 150.**

Let $A$ be an associative algebra. If $x, y, h = [x, y] \in A$ are three elements which satisfy the relations of $\mathfrak{sl}_2$, then $[x, y^t] = t y^{t-1}(h - t + 1)$.

Now we are ready to prove the first key proposition.

**Proposition 151.**

Let $\lambda, \mu \in \mathfrak{h}^*$, and $\alpha \in \Delta$ is simple. Write $n := \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}.$, so that $s_\alpha \cdot \lambda = \lambda - n\alpha$. Assume that $M(s_\alpha \cdot \mu) \subset M(\mu) \subset M(\lambda)$. Then:

1. If $n \leq 0$, then $M(\lambda) \subset M(s_\alpha \cdot \lambda)$.
2. If $n > 0$, then $M(s_\alpha \cdot \mu) \subset M(s_\alpha \cdot \lambda) \subset M(\lambda)$.
   Note that in either case, $M(s_\alpha \cdot \mu) \subset M(s_\alpha \cdot \lambda)$.

This proposition should be thought of as: if we know a bit about $M(\mu)$ and $M(\lambda)$, then we can extract information about $M(s_\alpha \cdot \mu)$ and $M(s_\alpha \cdot \lambda)$.

***Proof.***

1. This part is easy. Proposition 144 implies that $M(\lambda) \subset M(s_\alpha \cdot \lambda)$, since $n \leq 0$ so $\lambda \leq s_\alpha \cdot \lambda$ and they are related by a simple reflection.
2. Again, Proposition 144 implies that $M(s_\alpha \cdot \lambda) \subset M(\lambda)$. So it remains to show that $M(s_\alpha \cdot \mu) \subset M(s_\alpha \cdot \lambda)$. What we already know is that

$$\begin{array}{ccc} M(s_\alpha \cdot \lambda) & \subset & M(\lambda) \\ & & \cup \\ M(s_\alpha \cdot \mu) & \subset & M(\mu). \end{array}$$

   What we want to prove is that $M(s_\alpha \cdot \mu) \subset M(s_\alpha \cdot \lambda)$. The idea to prove this is to track the highest weight vectors generating these Verma modules. We have highest weight vectors $v_\lambda \in M(\lambda)$ and $y_\alpha^n v_\lambda \in M(s_\alpha \cdot \lambda)$; similarly, we have $v_\mu \in M(\mu)$ and $y_\alpha^r v_\mu \in M(s_\alpha \cdot \mu)$:

$$\begin{array}{ccccc} y_\alpha^n v_\lambda \in & M(s_\alpha \cdot \lambda) & \subset & M(\lambda) & \ni v_\lambda \\ & & & \cup & \\ y_\alpha^r v_\mu \in & M(s_\alpha \cdot \mu) & \subset & M(\mu) & \ni v_\mu. \end{array}$$

   Our goal is to show that $y_\alpha^r v_\mu \in M(s_\alpha \cdot \lambda)$ (understood by the embeddings above), which will then imply that $M(s_\alpha \cdot \mu) \subset M(s_\alpha \cdot \lambda)$.

Now since $M(\mu) \subset M(\lambda)$, we know that $v_\mu = u v_\lambda$ for some $u \in \mathcal{U}(\mathfrak{n}_-)$. Therefore by Lemma 146, there exists $t$ such that $t_\alpha^t u \in \mathcal{U}(\mathfrak{n}_-) y_\alpha^n$. This means that for some $t$, we have

$$y_\alpha^t v_\mu = y_\alpha^t u v_\lambda \in \mathcal{U}(\mathfrak{n}_-) y_\alpha^n v_\lambda \subset M(s_\alpha \cdot \lambda).$$

So for $t \gg 0$, we have $y_\alpha^t v_\mu \in M(s_\alpha \cdot \lambda)$. It only remains to check that we can take $t = r$. The idea is simple: since we can take arbitrarily large $t$, we may assume $t \geq r$, and then take smaller and smaller $t$. Namely, if $t > r$, then Lemma 147 implies that

$$
\begin{aligned}
[x_\alpha, y_\alpha^t]v_\mu &= t y_\alpha^{t-1}(\underbrace{h_\alpha}_{r-1} - t + 1)v_\mu, \\
\implies (x_\alpha y_\alpha^t - \underbrace{y_\alpha^t x_\alpha}_{\text{acts by } 0})v_\mu &= (r-t)t y_\alpha^{t-1} v_\mu, \\
\implies M(s_\alpha \cdot \lambda) \ni x_\alpha(\underbrace{y_\alpha^t v_\mu}_{\in M(s_\alpha \cdot \lambda)}) &= (r-t)t y_\alpha^{t-1} v_\mu.
\end{aligned}
$$

Now since $t > r$, then $(r-t)t \neq 0$, so $y_\alpha^{t-1} v_\mu \in M(s_\alpha \cdot \lambda)$. Therefore we can continue in this way until $t = r$, at which point we've proved our goal, that $y_\alpha^r v_\mu \in M(s_\alpha \cdot \lambda)$, which then implies that $M(s_\alpha \cdot \mu) \subset M(s_\alpha \cdot \lambda)$.

$\square$

This was actually the key fact for today's lecture!

Finally, let's prove Theorem 143 (Verma).

***Proof.***
First, we **assume that** $\lambda \in \Lambda$ **is integral**.
We know that there exists $w \in W$ such that $\mu' := w^{-1} \cdot \mu \in \Lambda^+ - \rho$. Let's write $w = s_n \cdots s_1$, and put $\mu_k := (s_k \cdots s_1) \cdot \mu'$. Also write $\lambda' := w^{-1} \cdot \lambda$ and $\lambda_k := (s_k \cdots s_1) \cdot \lambda'$. For convenience, denote $w_k := s_k \cdots s_1$. Note that we know plenty about $\mu_k$'s, but we don't know too much about $\lambda_k$'s.

We know that $\mu = s_\alpha \cdot \lambda$, which implies (easy exercise) that $\mu_k = s_{\beta_k} \cdot \lambda_k$, where $\beta_k = w_k^{-1}(s_\alpha)$. In particular, we know that $\mu_k = \lambda_k - \langle \lambda_k + \rho, \beta_k^\vee \rangle \beta_k$, which means that either $\lambda_k > \mu_k$ or $\mu_k > \lambda_k$. (Note that we don't know much about the relation between $\lambda_k$ and $\mu_k$, but we do know that they are directly comparable - because $\beta_k$ is just a single root!)

We have

$$
\mu = \mu_n \leq \mu_{n-1} \leq \cdots \leq \mu_{k+1} \leq \mu_k \leq \cdots \leq \mu_1 \leq \mu_0 = \mu'.
$$

We also have

$$
\lambda = \lambda_n \quad \lambda_{n-1} \quad \ldots \quad \lambda_{k+1} \quad \lambda_k \quad \ldots \quad \lambda_0 = \lambda'.
$$

However, unlike with the $\mu_i$'s, we don't know any relations between them. What we do know is the following comparisons (denoted by vertical $>$ and $<$): that $\mu_0 = \mu' > \lambda' = \lambda_0$, but $\mu_n = \mu < \lambda = \lambda_n$, and that the signs are consistent until some index where they flip.

$$\begin{array}{cccccccc}
\mu = \mu_n \ \leq & \mu_{n-1} \ \leq & \ldots \ \leq & \mu_{k+1} \ \leq & \mu_k \ \leq & \ldots \ \leq & \mu_1 \ \leq & \mu_0 = \mu' \\
\wedge & \wedge & \wedge & \wedge & \vee & \vee & \vee & \vee \\
\lambda = \lambda_n & \lambda_{n-1} & \ldots & \lambda_{k+1} & \lambda_k & \ldots & \lambda_1 & \lambda_0 = \lambda'
\end{array}$$

Say $k$ is exactly this index (where the sign flips), as in the above image, i.e. $k$ is minimal such that $\mu_k > \lambda_k$. Then the difference (note that $\cdot$ means $\rho$-shifted, no $\cdot$ means no $\rho$-shift) shows:

$$\underbrace{\mu_{k+1} - \lambda_{k+1}}_{\text{negative multiple of } \beta_{k+1}} = s_{k+1} \cdot \mu_k - s_{k+1} \cdot \lambda_k = s_{k+1} \underbrace{(\mu_k - \lambda_k)}_{\text{positive multiple of } \beta_k}.$$

But the only positive root that $s_{k+1}$ makes into a negative root is $\alpha_{k+1}$, hence

$$\beta_k = \beta_{k+1} = \alpha_{k+1}.$$

So $\mu_{k+1} = s_{\alpha_{k+1}} \cdot \lambda_{k+1} < \lambda_{k+1}$. Since $\alpha_{k+1}$ is not just any root, but it is actually a simple root, we can apply Proposition 144 to find that

$$M(\mu_{k+1}) \subset M(\lambda_{k+1}) \implies M(\mu_{k+2}) \subset M(\mu_{k+1}) \subset M(\lambda_{k+1}).$$

Now we apply Proposition 148 to see that

$$M(\mu_{k+2}) \subset M(\lambda_{k+2}),$$

and so on, eventually concluding

$$M(\mu) = M(\mu_n) \subset M(\lambda_n) = M(\lambda).$$

I'll only give a sketch of the proof for the general case, when $\lambda \in \mathfrak{h}^*$, however the full proof can be found in §4.7 of Humphreys' book. Fix $n \in \mathbb{Z}_{>0}$ and $\alpha \in \Phi^+$, and $H = H_{\alpha,n} = \{\tau \in \mathfrak{h}^* \mid \langle \tau + \rho, \alpha^\vee \rangle = n\}$. Let $X := \{\tau \in H \mid \exists M(s_\alpha \cdot \lambda) \subset M(\lambda)\} \subset H$. The goal is to show that in fact $X = H$. We know that $\Lambda \cap H \subset X$, and in fact we claim that $\Lambda \cap H$ is dense in $X$ in the Zariski topology (left as an exercise; if $p(x_1, \ldots, x_n)$ is a polynomial such that $p|_{\mathbb{Z}^n} = 0$, then $p = 0$; use induction on $n$). Then it remains to check that $X$ is closed in $H$ (in the Zariski topology), which will prove that $X = H$, proving the theorem.

□

# Nov 8

## Shapovalov determinant for $\widehat{\mathfrak{g}}$, with a focus on $\widehat{\mathfrak{sl}_2}$

Recall that in the finite-dimensional case, we have the Verma theorem:

**Theorem 152** (Verma)**.**

If $\mu < \lambda$ in the partial order on the roots, and $s_\alpha \cdot \lambda = \mu$ (for any $\alpha > 0$, not just simple roots), then we have an embedding $M(\lambda) \supset M(\mu)$.

The condition that $s_\alpha \cdot \lambda = \mu$ is equivalent to

$$\lambda - \mu = \langle \lambda + \rho, \alpha^\vee \rangle \cdot \alpha = \frac{2\langle \lambda + \rho, \alpha \rangle}{\langle \alpha, \alpha \rangle} \cdot \alpha.$$

**Corollary 153.**

Let $m = \frac{2\langle \lambda+\rho,\alpha \rangle}{\langle \alpha,\alpha \rangle}$. Then $\mathcal{D}_{m\alpha}(\lambda) = 0$, which is equivalent to $\mathcal{D}_{m\alpha}(\lambda)$ being divisible by $2\langle \lambda + \rho, \alpha \rangle - m\langle \alpha, \alpha \rangle$, which is a linear function of $\lambda$ plus some constant.

So we have obtained a new factor of the Shapovalov determinant for any multiple of any root. According to the general theorem, this is sufficient: this allows us to split the Shapovalov determinants into linear factors.

**Corollary 154.**

Any $\mathcal{D}_\nu(\lambda)$ splits into linear factors as in Corollary 150, corresponding to $m\alpha$ satisfying $m\alpha \leq \nu$.

This follows from the upper bound of the degrees of the determinants and the divisibility theorem (that any $\mathcal{D}_\nu$ is divisible by a sufficient power of the previous ones).

**Corollary 155.**

If $2\langle \lambda + \rho, \alpha \rangle - m\langle \alpha, \alpha \rangle \neq 0$ for all $\alpha > 0$ and $m > 0$, then $M(\lambda)$ is irreducible.

Note that in Theorem 149 (Verma), this is if and only if.

## Extending this to $\widehat{\mathfrak{g}}$

Recall that we have the decomposition

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{n}}_- \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}_+,$$

where $\widehat{\mathfrak{h}}$ is spanned by the usual Cartan $\mathfrak{h}$ and the central element $c$. We have the very similar decomposition

$$\widetilde{\mathfrak{g}} = \widehat{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}_+,$$

where the only difference is that the Cartan is upgraded to $\widetilde{\mathfrak{h}}$, which is spanned by $\mathfrak{h}, c, d$. Why is this better? It's because it gives an additional grading and it makes the weight spaces in $\widehat{\mathfrak{n}}_\pm$ finite-dimensional (otherwise, with respect to the usual Cartan $\mathfrak{h}$, the weight spaces are infinite-dimensional (and the central element $c$ acts trivially via adjoint)). So this addition of $d$ is actually crucial.

Important question: What are the roots?

## Roots

This is an infinite root system; there are infinitely many roots. The roots are determined by the following property:

$$\Phi = \left\{\alpha \in \widetilde{\mathfrak{h}} \mid \exists\, 0 \neq e_\alpha \in \widehat{\mathfrak{n}}_\pm \text{ s.t. } \forall x \in \widetilde{\mathfrak{h}},\ [x, e_\alpha] = \alpha(x) \cdot e_\alpha\right\}.$$

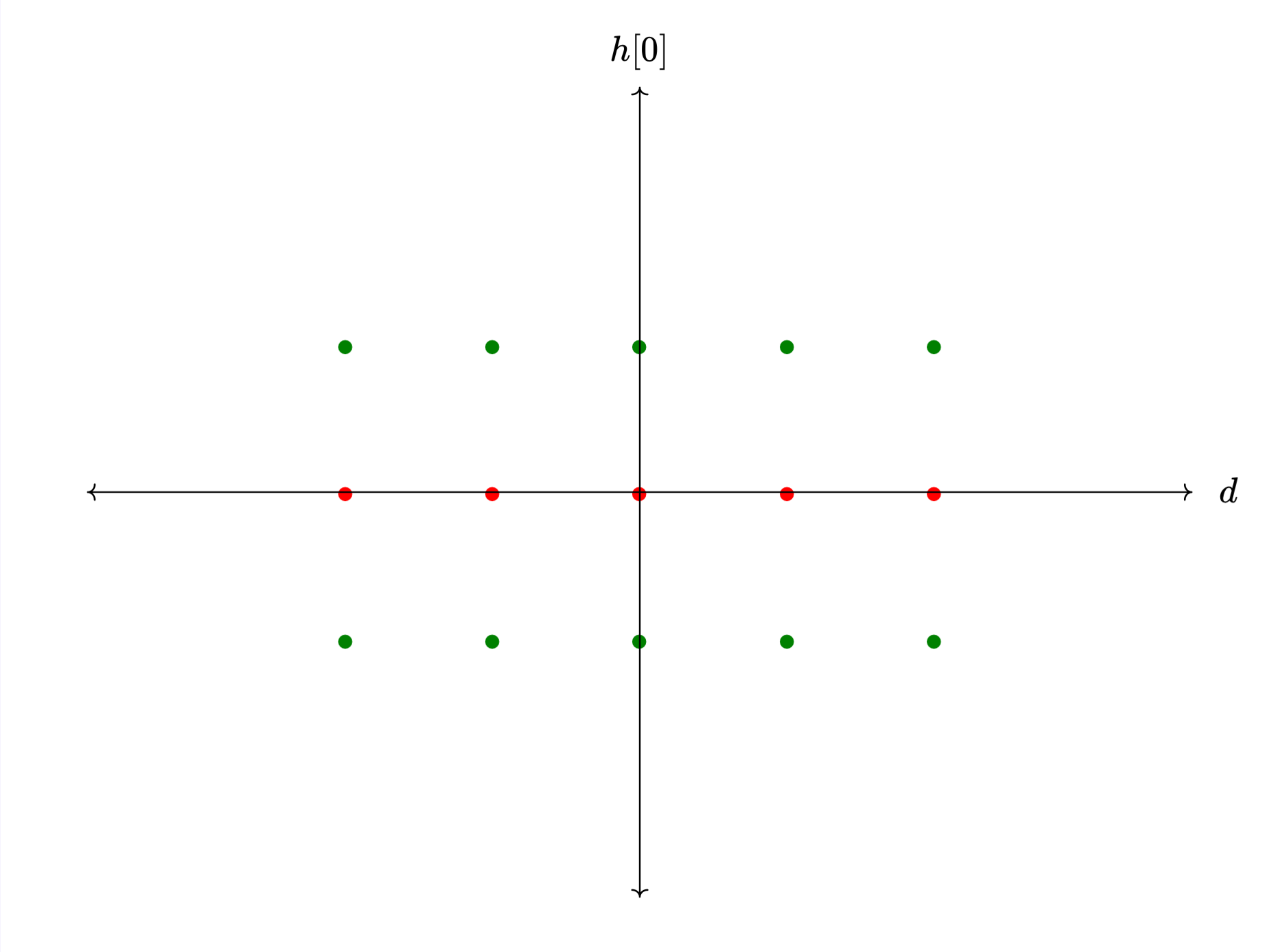
First, note that this is orthogonal to the central element $c$; if we substitute $c$, we get $0$. So all of the roots lie in a hyperplane. So let us denote $\{c\}^\perp =: \widehat{\mathfrak{h}}^\vee$.

We have a nondegenerate form on this Cartan $\widetilde{\mathfrak{h}}$ which pairs $c$ with $d$. So this means that the restriction to $\widehat{\mathfrak{h}}^\vee$ is degenerate ($c$ is an isotropic vector). So in particular this means the scalar product of any root with itself $\langle \alpha, \alpha \rangle \geq 0$ is always at least $0$, but strangely, it *can* be $0$. Therefore we usually think of roots, bilinear pairing, etc. only on $\widetilde{\mathfrak{g}}$ (rather than $\widehat{\mathfrak{g}}$).

**Example 156.**

Let $\mathfrak{g} = \mathfrak{sl}_2$. Then $\widehat{\mathfrak{sl}_2}$ splits into a direct sum of 1-dimensional root subspaces.
What are the roots? In the $(\operatorname{ad} h[0], \operatorname{ad} d)$-eigenvalue coordinates (i.e. bigrading from $h[0]$ and $d$), they are $\{-2, 0, 2\} \times \mathbb{Z}$, except for $(0, 0)$.
Now what is the scalar product? Well, it doesn't depend on the second coordinate (on the $d$-eigenvalue). It turns out to be $\langle (x_1, x_2), (y_1, y_2) \rangle = \frac{1}{2} x_1 y_1$. So we find that all of the roots with $d$-grading $\pm 2$ have $\langle \alpha, \alpha \rangle = 2$, we call them **real roots** (green in the below diagram). All of the roots

with $d$-grading 0 have $\langle \alpha, \alpha \rangle = 0$, we call them **imaginary roots** (red in the below diagram).




Now for arbitrary $\mathfrak{g}$, the roots are given by:

$$\underbrace{\Phi \times \mathbb{Z}}_{\text{real}} = \{(\alpha, n) \mid \alpha \in \Phi, n \in \mathbb{Z}\}, \qquad \underbrace{\{0\} \times \mathbb{Z}_{\neq 0}}_{\text{imaginary}} = \{(0, n) \mid 0 \neq n \in \mathbb{Z}\}.$$

For a real root $\alpha$, we can define the corresponding reflection

$$s_\alpha : \beta \mapsto \beta - \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \cdot \alpha = \beta - \langle \alpha^\vee, \beta \rangle \cdot \alpha, \quad \alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}.$$

We still have an $\mathfrak{sl}_2$-triple $e = e_\alpha$, $f = e_{-\alpha}$, and $h_\alpha := [e_\alpha, e_{-\alpha}]$ (the $h_\alpha$ corresponds to $\alpha^\vee$, thus giving an identification $\widetilde{\mathfrak{h}} \leftrightarrow \widetilde{\mathfrak{h}}^*$, just as in the finite-dimensional case).
The moral of the story is that we **still have roots**, and **the real roots behave just as in the finite-dimensional situation**; the other imaginary roots behave differently.

For imaginary roots $\alpha$, we have a Heisenberg subalgebra (though it is not uniquely determined, since they may not be multiplicity $1$, so you need to make a choice of a basis in $\mathfrak{h}$).

## What is $\rho$?

In all of those formulas, we had $\rho$. What is $\rho$? In the finite-dimensional case, we define $\rho := \frac{1}{2}\sum_{\alpha>0}\alpha$. Obviously this doesn't work here since there are infinitely many positive roots. But another definition is the sum of the fundamental weights. This definition will work in the infinite-dimensional case, but we need to define what a fundamental weight is.

The usual definition of fundamental weight is $\langle \omega_i, \alpha_i^\vee \rangle = \delta_{ij}$, i.e., dual pairing with the simple roots. We already discussed what the simple roots are in the affine situation.

**Simple roots:** these should be a "basis" so that anything in $\widehat{\mathfrak{n}}_+$ is a nonnegative integer linear combination of simple roots.

**Definition 157** (simple roots)**.**

Let $\alpha_1, \ldots, \alpha_r$ be the simple roots for $\mathfrak{g}$. For $\widehat{\mathfrak{g}}$, our **simple roots** will be

$$\underbrace{(\alpha_1, 0)}_{\widehat{\alpha_1}}, \underbrace{(\alpha_2, 0)}_{\widehat{\alpha_2}}, \ldots, \underbrace{(\alpha_r, 0)}_{\widehat{\alpha_r}}, \quad \underbrace{(-\theta, 1)}_{\widehat{\alpha_0}}$$

where $\theta$ is the highest root.

Once we have simple roots, we have simple coroots as well.

**Definition 158.**

The **simple coroots** are

$$\{(\alpha_i^\vee, 0)\}, \quad \frac{2(-\theta, 1)}{\langle \theta, \theta \rangle} = (-\theta^\vee, \frac{2}{\langle \theta, \theta \rangle}).$$

What about the fundamental weight $\widehat{\omega_i}$? Well, it should be a triple $(-,-,-) \in \mathfrak{h}^* \times \mathbb{C}c^* \times \mathbb{Z}$, but the $\mathbb{Z}$ component (regarding $d$) doesn't matte. The $\mathfrak{h}^*$ component is $\omega_i$, and the $\mathbb{C}c^*$ component is $\langle \omega_i, \theta^\vee \rangle$; this gives us an $\widehat{\omega_i}$ which pairs with $\alpha_i^\vee$ to $0$. On the other hand, $\widehat{\omega_0} = (0, 1)$. This means that we can define $\widehat{\rho}$:

**Definition 159** ($\widehat{\rho}$)**.**

$$\widehat{\rho} = \sum_{i=0}^{r} \widehat{\omega_i} = (\rho, 1 + \langle \rho, \theta^\vee \rangle).$$

Note that $\langle \rho, \theta^\vee \rangle$ is just a number depending on $\mathfrak{g}$.

**Example 160.**

Let $\mathfrak{g} = \mathfrak{sl}_n$. Then $1 + \langle \rho, \theta^\vee \rangle = 1 + (n-1) = n$.

**Remark 161.**

The number $\langle \rho, \theta^\vee \rangle$ is called the (dual) Coxeter number, and there are several definitions of it.

Now we're trying to generalize Corollary 150 to $\widehat{\mathfrak{g}}$. For $\alpha$ a real root, there is no problem; all of the linear factors are different, and they all contribute to the determinant of the Shapovalov form. But when $\alpha$ is imaginary, we get problems: then $m\langle \alpha, \alpha \rangle = 0$, so it somehow doesn't depend on $m$. Then all of the factors coincide, so we can't just say that the Shapovalov form is divisible by the product of these factors. So we need to do something more.

For next time, we'll compute $\mathcal{D}_\nu(\lambda)$ for $\widehat{\mathfrak{sl}_2}$; we just need to investigate what happens at the imaginary roots.

# Nov 13

## Singular vectors in a Verma module (with focus on $\widehat{\mathfrak{sl}_2}$)

Let's recall what we already know.
We have an affine root system, on the extended affine Lie algebra

$$\widetilde{\mathfrak{g}} = \widehat{\mathfrak{n}}_- \oplus \underbrace{\mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d}_{\widetilde{\mathfrak{h}}} \oplus \widehat{\mathfrak{n}}_+.$$

We do this in order to make the invariant scalar product nondegenerate on the Cartan subalgebra. Then we consider $\widetilde{\mathfrak{h}}^*$; we have the roots $\widehat{\Phi}$ satisfying

$$\widetilde{\mathfrak{h}}^* \supset (\mathbb{C}c)^\perp \supset \widehat{\Phi} = \left\{ \underbrace{(\alpha + nd^\vee)}_{e_\alpha \cdot z^n}, \underbrace{(0 + \overbrace{n}^{n \neq 0} d^\vee)}_{h \cdot z^n} \mid \alpha \in \Phi, n \in \mathbb{Z} \right\}.$$

Here, $\Phi$ denotes the roots of $\mathfrak{g}$. Note that the root system lies in the codimension $1$ subspace orthogonal to the central element $c$, as $\operatorname{ad} c$ acts by $0$. Inside the roots $\widehat{\Phi}$ we have the subset of simple roots $\widehat{\Pi}$:

$$\widehat{\Pi} = \begin{cases} \alpha_i & \alpha_i \in \Pi, \text{ simple roots for } \mathfrak{g} \\ -\theta + d^\vee =: \alpha_0 & \theta \text{ highest root for } \mathfrak{g}. \end{cases}$$

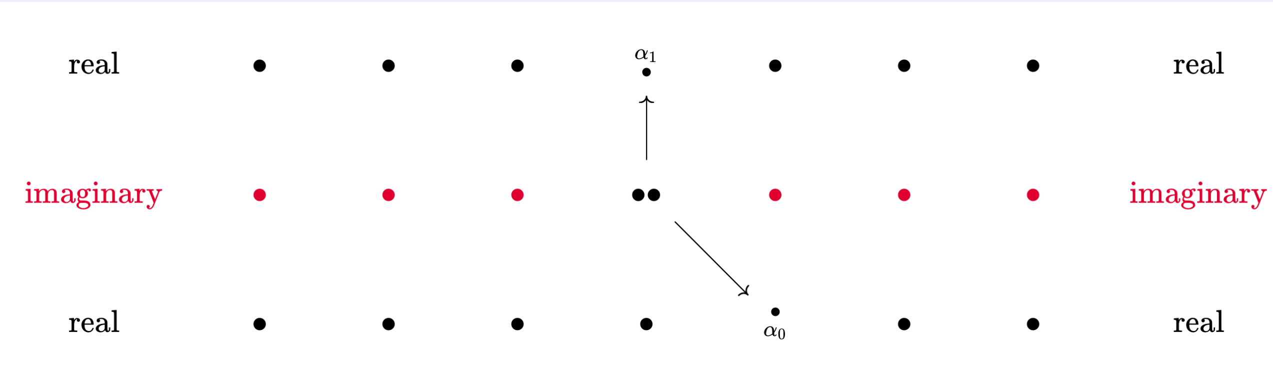
We also have the weight lattice

$$\widehat{\Lambda} = \left\{ \widehat{\lambda} \in \widehat{\mathfrak{h}}^* \mid \langle \widehat{\lambda}, \alpha^\vee \rangle \in \mathbb{Z} \, \forall \, \text{simple roots } \alpha \right\}.$$

Note that if $\widehat{\lambda} \in \widehat{\Lambda}$, then so is $\widehat{\lambda} + rd^\vee$, so in fact we may assume that $\widehat{\Lambda} \subset (\mathbb{C}d)^\perp$.

**Example 162.**

Let's consider $\widehat{\mathfrak{sl}_2}$. The root system is as follows.




Next, we need to understand the **affine Weyl group**.

In the finite-dimensional situation we consider the group generated by reflections $s_\alpha$ for all simple roots $\alpha$. We basically do the same thing:

**Definition 163** (affine Weyl group).

The **affine Weyl group** $\widehat{W}$ is the group generated by reflections $s_\alpha$ for $\alpha \in \widehat{\Pi}$, where

$$s_\alpha : \beta \mapsto \beta - \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \alpha.$$

The affine Weyl group acts on $\widehat{\mathfrak{h}}^*$, preserving $\widehat{\Phi}$.

**Example 164.**

Let's consider $\widehat{\mathfrak{sl}_2}$. Recall that $\mathfrak{sl}_2$ has a single root $\alpha = 2$; note that $\langle \alpha, \alpha \rangle = 2$ so in fact $\alpha = \alpha^\vee$ for any real root.. Let $\widehat{\lambda} \in \widetilde{\mathfrak{h}}^*$, so that $\widehat{\lambda} = \lambda\alpha + \varepsilon c^\vee + \eta d^\vee$. Then $\widehat{\mathfrak{sl}_2}$ has two simple roots, $\alpha_1 = \alpha$ and $\alpha_0 = -\alpha + d^\vee$. Let's see what $s_{\alpha_1}$ does. We can compute that

$$s_{\alpha_1}(\widehat{\lambda}) = -\lambda\alpha + \varepsilon c^\vee + \eta d^\vee.$$

On the other hand, we have $\alpha_0 = -\alpha + d^\vee = -2 + d^\vee$, so

$$s_{\alpha_0}(\widehat{\lambda}) = \lambda\alpha_1 + \varepsilon c^\vee + \eta d^\vee - (-2\lambda + \varepsilon) \cdot (-\alpha_1 + d^\vee) = (\varepsilon - \lambda)\alpha_1 + \varepsilon c^\vee + (\eta - \varepsilon + 2\lambda) d^\vee.$$

So in this case, $\widehat{W}$ is freely generated by $s_{\alpha_1}$ and $s_{\alpha_0}$, which are reflections of the affine line $\mathbb{A}^1$ with two different centers.

**Observation:** $\widehat{W}$ is generated by $s_\alpha$ for $\alpha$ real. (This is clear for this case, but in fact it's true in general).

**Remark 165.**

In fact, the real roots are precisely the roots which can be obtained from a simple root using the action of the (affine) Weyl group.

## Shifted action of $\widehat{W}$

We have the analogue of $\rho$:

$$\widehat{\rho} := \sum \widehat{\omega}_i.$$

**Example 166.**

For $\widehat{\mathfrak{sl}_2}$, we have $\widehat{\rho} = \frac{\alpha_1}{2} + 2c^\vee$.

**Definition 167.**

The **dot action** of $\widehat{W}$ on $\widetilde{\mathfrak{h}}^*$ is

$$w \cdot \lambda := w(\widehat{\lambda} + \widehat{\rho}) - \widehat{\rho}.$$

On Nov 6 we proved Theorem 143 (Verma) for finite-dimensional Lie algebras. Here is the analogue for affine Lie algebras.

**Theorem 168** (Verma)**.**

Suppose $\alpha$ is a real root, and $\widehat{\lambda}$ is such that $s_\alpha \cdot \widehat{\lambda} < \widehat{\lambda} \iff \frac{2\langle \widehat{\lambda} + \widehat{\rho}, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}_{>0}$. Then there is an embedding $M(s_\alpha \cdot \widehat{\lambda}) \subset M(\widehat{\lambda})$.

***Proof.***
We want to show that the plan of the proof of Theorem 143 (Verma) works here. Let's recall the

plan.

1. $M(s_\alpha \cdot \lambda) \subset M(\lambda)$ if $\alpha$ is simple.

The argument here is actually the same. To any real root $\alpha$ corresponds an $\mathfrak{sl}_2$-triple $\{e_\alpha, h_\alpha, f_\alpha\}$. Now say $m = \frac{2\langle\widehat{\lambda}+\widehat{\rho},\alpha\rangle}{\langle\alpha,\alpha\rangle}$. Then $f_\alpha^m v_{\widehat{\lambda}}$ is a singular vector, since $e_\alpha$ acts by zero, and all of the other raising operators act by zero as well.
2) It was sufficient to prove for integral $\lambda$, because then we apply some interpolation argument.

In our case, it's sufficient to prove for integral $\widehat{\lambda}$ and the level $k$ satisfies $k > -2$, because then there exists $w \in \widehat{W}$ with $w \cdot \widehat{\lambda} + \widehat{\rho}$ is dominant (i.e., scalar product with any simple coroot is nonnegative). This means that if we take such a weight, then for any simple reflection $s_\alpha$ acting on the highest weight vector of this weight, we have a submodule.
3) Once we have this, we only need the following lemma.

**Lemma 169.**

Assume that $M(s_\alpha \cdot \widehat{\mu}) \subset M(\widehat{\mu}) \subset M(\widehat{\lambda})$. Then $M(s_\alpha \cdot \widehat{\mu}) \subset M(s_\alpha \cdot \widehat{\lambda})$.

The proof is actually the same as in the finite-dimensional case (see proof of Proposition 148). It uses only the nilpotence of $\operatorname{ad} f_\alpha$, which will imply local nilpotence of $\operatorname{ad} f_\alpha$ on the universal enveloping algebra $\mathcal{U}(\widehat{\mathfrak{n}}_-)$.

□

# Nov 15

## $\widehat{\mathfrak{sl}_2}$ Verma modules on the level $k = -2$

**Observation:** the factors in the Shapovalov determinant $\mathcal{D}_\nu(\Lambda)$ are always of the following form: $2\langle\Lambda + \rho, \alpha\rangle - n \cdot \langle\alpha, \alpha\rangle$ for some $n \in \mathbb{Z}_{>0}$ and $\alpha$ some positive root.

For real roots, we have seen how these factors arise: once this condition is satisfied, we can construct a singular vector in the Verma module. This formula is really nice, so we might suspect that it holds for *any* root, whether real or imaginary. Let us see what this means for imaginary roots. For imaginary $\alpha$, this factor is $2(c+2) - 0$ (where $c$ is the central element). So we expect something special happening at the level $k = -2$. So let's see where this comes from. We'll need a recollection of Sugawara elements (in the completed universal enveloping algebra).

### Sugawara elements

Recall that the Sugawara elements come from trying to construct central elements in the loop algebra; we do this by using the formula for the Casimir element and substituting the relevant elements corresponding to the generators of $\mathfrak{sl}_2$.

Define $e(u) = \sum e[r]e^{-r-1}$, and we define $f(u)$, $h(u)$ similarly.
Then define the Sugawara element power series to be

$$\begin{aligned} \mathbb{S}(u) &= \quad : e(u)f(u) : \quad + \quad : f(u)e(u) : \quad + \quad : \frac{1}{2}h(u)h(u) : \\ &= \sum_n S_n u^{-n-2}, \\ S_n &= \sum_{r+s=n, \quad s>0} e[r]f[s] + f[r]e[s] + \frac{1}{2}h[r]h[s] + \sum_{r+s=n, \quad s\leq 0} f[s]e[r] + e[s]f[r] + \frac{1}{2}h[s]h[r]. \end{aligned}$$

What is good about these elements $S_n$ is that the commutator $[S_n, -]$ acts as an element of the Witt algebra. We have an action $[S_n, -] \curvearrowright \mathcal{U}(\widehat{\mathfrak{g}})_k = \mathcal{U}(\widehat{\mathfrak{g}})/(c-k)$. Of course, we can compute it explicitly.

**Proposition 170.**

$[S_n, -] = 2(c+2)z^{n+1}\partial_z$.

***Proof.***
This can be checked directly via computation, but we can optimize it as follows.
First, the quadratic term in $[S_n, x[r]]$ vanishes, so $[S_n, -]$ comes from a derivation of $\widehat{\mathfrak{g}}$.
Second, $[S_n, -]$ commutes with $\mathfrak{g} \subset \widehat{\mathfrak{g}}$, because it was constructed from the Casimir element, which is $\mathfrak{g}$-invariant. (These two steps are general.)
Third, we know that $\mathcal{D}\mathrm{er}\,\widehat{\mathfrak{g}}/\mathrm{ad}\,\widehat{\mathfrak{g}} \cong W$, the Witt algebra. This means that $[S_n, -]$ acts as some element of $W$.
Finally, $S_n$ is homogeneous degree $n$ with respect to $z\partial_z$, so $[S_n, -]$ is a constant times $z^{n+1}\partial_z$.
Therefore it suffices to compute the coefficient, which we can find by computing
$[S_n, h[1]] = 2(c+2)h[n+1]$.

□

**Corollary 171.**

When $k = -2$, all of the $S_n$ are central.

**Corollary 172.**

In $M(\lambda, -2)$ we have many singular vectors: for example, $\prod_{n<0} S_n^{m_n} \cdot v_{\lambda,-2}$ is singular.

**Proposition 173.**

All of these singular vectors are linearly independent.

It's possible to prove this in several different ways. First, a weaker version:

**Lemma 174.**

For any $k \in \mathbb{C}$ and Weyl-generic $\lambda$, then all monomials $\prod_{r<0} e[r]^{k_r} \prod_{s \le 0} f[s]^{l_s} \prod_{n<0} S_n^{m_n} \cdot v_{\lambda,k}$ forms a basis of $M_{\lambda,k}$. (Note that the $S_n$ do not commute with each other! But that's ok; choose some arbitrary order, as we did here.)

***Proof.***
Take $\lambda \to \infty$, then the leading term in $S_n$ comes from $h[0] \cdot h[n]$, which is just $h[n]$, hence the leading term is a PBW monomial in the PBW basis $\underbrace{e[r]}_{r<0}, \underbrace{f[s]}_{s\le 0}, \underbrace{h[n]}_{n<0} \in \widehat{n}_-$, which is known to be a basis in $M(\lambda, k)$.

$\square$

The stronger version:

**Lemma 175.**

Consider the images of all Sugawara elements

$$\underbrace{Z\left(\widetilde{\mathcal{U}(\widehat{\mathfrak{sl}_2})_{-2}}\right)}_{\text{center}} \to \left(\mathcal{U}(\widehat{\mathfrak{sl}_2})/\mathcal{U}(\widehat{\mathfrak{sl}_2}) \cdot (\widehat{\mathfrak{n}}_+, h[0] - \lambda, c+2)\right)^{\widehat{\mathfrak{b}}_+} \hookrightarrow \mathcal{U}(\widehat{\mathfrak{n}}_-).$$

(Note that this is the same as using the completion, since we are quotient. Also, the quotient is just a module, but the $\widehat{\mathfrak{b}}_+$-invariants form an algebra.)
Then suppose that $S_n \mapsto \overline{S_n}(\lambda)$ under the composite of these maps; for $n \ge 0$, then $S_n \mapsto 0$. The claim is that for $n < 0$, the $\overline{S_n}(\lambda)$ stay algebraically independent.

***Proof.***
For $n < 0$ we can consider $\operatorname{gr}_{\mathrm{PBW}} \overline{S_n}(\lambda) \in S(\widehat{\mathfrak{n}}_-)$ (associated graded with respect to PBW). The formula is given by

$$\operatorname{gr}_{\mathrm{PBW}} \overline{S_n}(\lambda) = \sum_{\substack{r+s=n \\ r<0,\, s\le 0}} \underbrace{e[r]f[s]}_{\text{survives}} + f[r]e[s] + \frac{1}{2} h[r]h[s] \Big/ e[0] = h[0] = 0.$$

We can prove this by taking differentials at each point and checking that the differentials are linearly independent:

$$d_x \mathrm{gr}_{\mathrm{PBW}} \overline{S_n}(\lambda)\Big|_{f[0]=1, \quad e[r]=f[r]=h[r]=0,\, r<0}.$$

Then the only term that survives in $S_n$ is $e[n]$ (see the underbrace above). The $e[n]$ are linearly independent; this means that the differentials of the $\overline{S_n}(\lambda)$ are linearly independent at some point, hence the $\overline{S_n}(\lambda)$ are algebraically independent overall.

□

To summarize, we reduced the problem to showing algebraic independent of certain polynomials, and the natural way to prove that is to compute their differentials (at some chosen point) and check that they're linearly independent. So from this, we see the size of the space of singular vectors in a Verma module at the critical level $k = -2$.

**Corollary 176.**

$\mathcal{D}_\nu(\lambda)$ is divisible by $(c+2)^E$ where

$$E = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P(\nu - mn \cdot \underbrace{\lambda^\vee}_{\text{all positive imaginary roots}}),$$

where $P(\mu)$ is the Kostant partition function, i.e. the number of ways to write $\mu = \sum_{\alpha>0} n_\alpha \alpha$ (note that Kostant partition function has a generating formula $\prod_{\alpha>0} \frac{1}{1-t^\alpha}$).

**Theorem 177** (Kac-Kazhdan)**.**

$$\mathcal{D}_\nu(\Lambda) = \prod_{\alpha \in \Phi_+} \prod_{n=1}^{\infty} (2\langle \lambda + \rho, \alpha \rangle - n \cdot \langle \alpha, \alpha \rangle).$$

We'll start with this theorem next time.

# Nov 17

## Characters of $\widehat{\mathfrak{g}}$-modules in category-$\mathcal{O}$

Let $N$ be a category-$\mathcal{O}$ module over $\widetilde{\mathfrak{g}}$. (We consider the extended Lie algebra $\widetilde{\mathfrak{g}}$ because we want to keep the grading with respect to $d$; without this grading, the weight spaces are infinite-dimensional, and the characters would not be well-defined. With $d$, the weight spaces are finite-dimensional, making the characters well-defined.)

**Definition 178.**

We define the **character** of $N$ to be

$$\operatorname{ch} N := \sum_{\nu \text{ weights of } N} t^{\nu} \dim N_{\nu}.$$

We know these characters when $N = M(\widehat{\lambda})$ is a Verma module. Then

$$\operatorname{ch} M(\widehat{\lambda}) = t^{\widehat{\lambda}} \cdot \prod_{\alpha \in \widehat{\Phi}_+} \frac{1}{1 - t^{-\alpha}}.$$

If we want to know the character of any category-$\mathcal{O}$ module, then by Jordan-Holder, we need to know the character of every simple module. So what we **want** is the characters of $L(\widehat{\lambda})$, the simple quotients of $M(\widehat{\lambda})$.

**Example 179.**

For Weyl-generic $\widehat{\lambda}$, then $L(\widehat{\lambda}) = M(\widehat{\lambda})$, so we know the character in this situation.

**Example 180.**

In the codimension 1 situation, let $\widehat{\lambda}$ be generic with the property $2\langle \widehat{\lambda} + \widehat{\rho}, \alpha \rangle - n \langle \alpha, \alpha \rangle = 0$. As we know from the determinant of the Shapovalov form, the Verma module becomes reducible iff this equation is satisfied for some root $\alpha$ and some $n$. Then depending on whether the root is real or imaginary, we can define the corresponding irreducible module.

**Proposition 181.**

Suppose we have $\widehat{\lambda}$ satisfying $2\langle \widehat{\lambda} + \widehat{\rho}, \alpha \rangle - n \langle \alpha, \alpha \rangle = 0$ for some **real** $\alpha$ and nonnegative integer $n$. There exists a unique submodule $M(s_\alpha \cdot \widehat{\lambda}) \subset M(\widehat{\lambda})$, and $L(\widehat{\lambda}) = M(\widehat{\lambda}) / M(s_\alpha \cdot \widehat{\lambda})$. As a corollary, the character of $L(\widehat{\lambda})$ is just the difference of the characters of the Verma modules, so we get

$$\operatorname{ch} L(\widehat{\lambda}) = \frac{t^{\widehat{\lambda}} - t^{s_\alpha \cdot \widehat{\lambda}}}{\prod_{\alpha \in \widehat{\Phi}_+} (1 - t^{-\alpha})}.$$

***Proof.***

$M(s_\alpha \cdot \widehat{\lambda})$ is the radical of the Shapovalov form (which is nondegenerate when restricted to this submodule), and this is the maximal proper submodule.

□

In this case, the computation of the character is easy. In general, it is much harder, and we have the Kac-Kazhdan conjecture:

**Conjecture 182** (Kac-Kazhdan)**.**

Suppose $\widehat{\lambda}$ is generic with $2\langle\widehat{\lambda}+\widehat{\rho},\alpha\rangle = 0$ for imaginary $\alpha$ ( $\iff$ the level $k = -h^\vee$ the Coxeter number, and $\lambda$ is generic). Then

$$\operatorname{ch} L(\widehat{\lambda}) = t^{\widehat{\lambda}} \cdot \prod_{\alpha\in\widehat{\Phi}_+^{\text{real}}} \frac{1}{1-t^{-\alpha}}.$$

This means that we can mod out by all of the imaginary roots.

In fact this is a theorem, but very difficult! It was first proved for $\widehat{\mathfrak{sl}_2}$ by Wakimoto, and the construction from the last homework proves it: you can construct some irreducible modules for generic $\widehat{\lambda}$ with precisely the character given above. For general $\widehat{\mathfrak{g}}$ it was proven by Feigen-Frenkel, by generalizing the construction of Wakimoto modules to the general case.

Now let's switch to some examples which are in some sense opposite.

## Integrable $\widehat{\mathfrak{g}}$-modules and their characters

Why are we interested in these? In the finite-dimensional case, in general, computing the character of $L(\lambda)$ is difficult: it involves some Kazhdan-Lusztig conjecture. But as we saw, for $\lambda$ close to being generic, we can compute the character. The other case where it's possible to compute the character is the opposite scenario, when $\widehat{\lambda}$ is dominant and integral, with the Weyl formula. In the case of affine Kac-Moody Lie algebras, the situation is similar.

First, we need to know: what is an integrable module? For $\widehat{\mathfrak{g}}$, integrable modules are the infinite-dimensional analogue of finite-dimensional irreducible modules.

**Definition 183** (integrable module)**.**

Let $N \in \mathcal{O}(\widetilde{\mathfrak{g}})$ be a category-$\mathcal{O}$ module. Then $N$ is called **integrable** if it is integrable with respect to any $\mathfrak{sl}_2$ formed by a simple root $\alpha$ (namely, with respect to the Lie algebra $\operatorname{Span}\{e_\alpha, f_\alpha, h_\alpha\}$).

In particular, we have the $\mathfrak{sl}_2$ triples formed by the simple roots for $\widetilde{\mathfrak{g}}$ coming from $\mathfrak{g}$, but then we have one more: we also need to check the $\mathfrak{sl}_2$-triple $\operatorname{Span}\{tf_\theta, c-h_\theta, t^{-1}e_\theta\}$.

This naturally leads us to our next question. **Which of the $L(\widehat{\lambda})$ are integrable?**

We have a necessary condition: the highest weight $\widehat{\lambda}$ must be dominant and integral with respect to any of these $\mathfrak{sl}_2$. In other words,

$$\frac{2\langle\widehat{\lambda},\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}_{\geq 0} \text{ for all simple } \alpha.$$

This is already very restrictive. Explicitly: we can assume $d$ acts by $0$ on the highest weight, so $\widehat{\lambda} = (\lambda, k, 0)$ can be expressed via coordinates via the action on $\mathfrak{h}$, the level (action of $\mathbb{C}c$), and $\mathbb{C}d$ (set to $0$ here). Then the necessary condition implies that $\lambda$ is dominant integral, i.e. $\frac{2\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}_{\geq 0}$ for all simple roots $\alpha$ of $\mathfrak{g}$, and $k \in \mathbb{Z}$. Finally we need to check what conditions the affine root $\theta$ imposes. What we get is (from the fact that the Cartan representative of that $\mathfrak{sl}_2$-triple is $c - h_\theta$) the condition $\frac{2\langle\lambda,\theta\rangle}{\langle\theta,\theta\rangle} \leq k$.

**Example 184.**

If $k = 0$, there is only possibility: $\lambda = 0 \implies \widehat{\lambda} = 0$, so the only integrable module is the trivial one, $L(\hat{0})$.

**Example 185.**

If $k = 1$, then in the $\widehat{\mathfrak{sl}_2}$ case, we have exactly two possibilities: $L(0,1)$ and $L(1,1)$.

**Proposition 186.**

The necessary condition is also sufficient.
In other words, $L(\widehat{\lambda})$ is integrable iff, when writing $\widehat{\lambda} = (\lambda, k, 0)$, the following two conditions are satisfied:

- for all simple roots $\alpha$ of $\mathfrak{g}$, that $\frac{2\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}_{\geq 0}$,
- $\frac{2\langle\lambda,\theta\rangle}{\langle\theta,\theta\rangle} \leq k$.

***Proof.***
We know that $L(\widehat{\lambda})$ is a quotient of $M(\widehat{\lambda})$. Fix $\alpha$ simple. Then we have a singular vector inducing the embedding $M(s_\alpha \cdot \widehat{\lambda}) \subset M(\widehat{\lambda})$. Therefore the surjection $M(\widehat{\lambda}) \twoheadrightarrow L(\widehat{\lambda})$ factors through the quotient $M(\widehat{\lambda}) \twoheadrightarrow M(\widehat{\lambda})/M(s_\alpha \cdot \widehat{\lambda})$.

We want to show that $M(\widehat{\lambda})/M(s_\alpha \cdot \widehat{\lambda})$ is already integrable with respect to the $\mathfrak{sl}_2$ corresponding to $\alpha$. But the space generated by this $\mathfrak{sl}_2$ from the highest vector $v_{\widehat{\lambda}}$ is finite-dimensional. So we have the space

$$\mathcal{U}(\mathfrak{sl}_2)v_{\widehat{\lambda}}/f_\alpha^m v_{\widehat{\lambda}}$$

for some power $m$; this is a finite-dimensional $\mathfrak{sl}_2$-module. Now everything can be obtained from the $\widehat{\mathfrak{g}}$-action, so we have a surjection of $\mathfrak{sl}_2$-modules

$$\mathcal{U}(\widehat{\mathfrak{g}}) \otimes \left(\mathcal{U}(\mathfrak{sl}_2)v_{\widehat{\lambda}}/f_\alpha^m v_{\widehat{\lambda}}\right) \twoheadrightarrow M(\widehat{\lambda})/M(s_\alpha \cdot \widehat{\lambda}).$$

Now the quotient (right side of tensor) is a finite-dimensional $\mathfrak{sl}_2$-module, while the $\mathcal{U}(\widehat{\mathfrak{g}})$ is a direct sum of finite-dimensional $\mathfrak{sl}_2$-modules. (This is because $\widehat{\mathfrak{g}}$ is itself a direct sum of finite-dimensional $\mathfrak{g}$-modules, hence is a direct sum of finite-dimensional $\mathfrak{sl}_2$-modules, with respect to the adjoint action.) So this tensor product is integrable, hence the image $M(\widehat{\lambda})/M(s_\alpha \cdot \widehat{\lambda})$ is also integrable, and finally the image of the surjection onto $L(\widehat{\lambda})$ is also integrable, proving the result.

□

This means that we have a classification of integrable irreducible $\widehat{\mathfrak{g}}$-modules, and it is the same as in the finite-dimensional case. We just want to compute the characters.

## Computing $\operatorname{ch} L(\widehat{\lambda})$

We want to express $\operatorname{ch} L(\widehat{\lambda})$ as a linear combination of $\operatorname{ch} M(\widehat{\mu})$. However, we don't have the finite length property, so we have to hope that we can represent $\operatorname{ch} \widehat{L(\lambda)}$ as an infinite sum of the $\operatorname{ch} M(\widehat{\mu})$. Fortunately, this turns out to be possible; not only that, it makes sense, because all of the Verma modules have a grading with respect to operator $\operatorname{ad} d$, so the character is a power series in several variables, one of which accounts for the eigenvalues of $d$. Let's use the variable $q$ to keep track of the grading with respect to $d$. (In this way, the $-r$ graded component with respect to $d$ will constitute the coefficient of $q^r$.)

The basic idea is as follows. If we have two submodules $M(\widehat{\mu}_1)$ and $M(\widehat{\mu}_2)$ inside $L(\widehat{\lambda})$, then first we mod out $M(\widehat{\mu}_1)$, which corresponds to subtracting $\operatorname{ch} M(\widehat{\mu}_1)$ from $\operatorname{ch} L(\widehat{\lambda})$. Then we want to mod out by the image of $M(\widehat{\mu}_2)$. But this is not actually subtracting $\operatorname{ch} M(\widehat{\mu}_2)$, because $M(\widehat{\mu}_1)$ and $M(\widehat{\mu}_2)$ intersect nontrivially; so we need to also add back in the intersection. The intersection has a highest weight vector, which then generates another Verma submodule, and so on.

After infinitely many steps, we will get $\operatorname{ch} L(\widehat{\lambda})$ as an infinite sum of characters of Verma modules, but whose coefficients are formal power series in $q$. This is actually well-defined!

Why are integrable modules easier than generic ones? This is because any integrable module $L(\widehat{\lambda})$ has an action of $\mathsf{SL}_2$ corresponding to any simple root $\alpha$. We can take

$$\tilde{s}_\alpha := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathsf{SL}_2 \curvearrowright L(\widehat{\lambda}).$$

This is an invertible operator which takes a weight space corresponding to $\widehat{\mu}$ to a weight space corresponding to $s_\alpha(\widehat{\mu})$. In other words, we have a map

$$\tilde{s}_\alpha : L(\widehat{\lambda})_{\widehat{\mu}} \to L(\widehat{\lambda})_{s_\alpha(\widehat{\mu})}.$$

This means that the character is stable with respect to the affine Weyl group:

**Proposition 187.**

Suppose $L(\widehat{\lambda})$ is integrable. Then $\operatorname{ch} L(\widehat{\lambda})$ is stable with respect to $\widehat{W}$, generated by all simple reflections $s_\alpha$ (for $\alpha$ a simple root).

Combining these two properties of the characters, we find:

**Proposition 188.**

Suppose $L(\widehat{\lambda})$ is integrable. Then

$$\operatorname{ch} L(\widehat{\lambda}) = \frac{\sum_{\text{dominant } \widehat{\mu}} \sum_{w \in \widehat{W}} a_{\widehat{\mu}} (-1)^w t^{w \cdot \widehat{\mu}}}{\prod_{\alpha \in \widehat{\Phi}_+} (1 - t^{-\alpha})},$$

where the product in the denominator is counted with multiplicity. Furthermore, $a_\lambda = 1$.

***Proof.***
We know that it's a linear combination of Verma modules, hence it must be of this form for some coefficients $a_{\widehat{\mu}}$. On the other hand, by Proposition 182, the character is stable under $\widehat{W}$. Note that on the denominator, $s_\alpha$ pretty much preserves the product except it swaps $-\alpha \mapsto \alpha$, hence it sends $(1 - t^{-\alpha}) \mapsto 1 - t^\alpha = -t^\alpha(1 - t^{-\alpha})$. To compensate for this, the coefficients of $t^\nu$ and $t^{s_\alpha \cdot \nu}$ should be negatives of each other; more generally, the coefficients of $t^\nu$ and $t^{w \cdot \nu}$ should differ by a sign of $(-1)^w$. So the character must be of the form above.

The last statement is clear since the weight space of $\widehat{\lambda}$ is one-dimensional; it's true in $M(\widehat{\lambda})$, and $L(\widehat{\lambda})$ is a quotient of the Verma module.

□

**Theorem 189** (Weyl-Kac formula)**.**

In Proposition 183, $a_{\widehat{\mu}} = 0$ for all $\widehat{\mu} \neq \widehat{\lambda}$.
In other words, if $L(\widehat{\lambda})$ is integrable, then

$$\operatorname{ch} L(\widehat{\lambda}) = \frac{\sum_{w\in\widehat{W}}(-1)^w t^{w\cdot\widehat{\lambda}}}{\prod_{\alpha\in\widehat{\Phi}_+}(1-t^{-\alpha})},$$

where the product in the denominator is counted with multiplicity.

**Remark 190.**

By "with multiplicity," we mean that $\alpha$ is counted $\dim\widehat{\mathfrak{g}}_\alpha$ number of times. In the finite-dimensional case, each subspace $\mathfrak{g}_\alpha$ has dimension $1$, but this is not true in the infinite-dimensional case.

The idea is to use the Casimir operator, which prohibits all other Verma modules except those in the $(\widehat{W},\cdot)$-orbit of $\widehat{\lambda}$.

For finite-dimensional $\mathfrak{g}$, this is really easy, because there are many Casimir elements, and computing the eigenvalues of all the central elements, we get many conditions (the eigenvalues have to be the same on all Verma modules), but it's known that the eigenvalues are only the same for those weights in the orbit of the Weyl group. In our infinite-dimensional case, we don't have so many central elements: we have only the one quadratic element, but it turns out this is still sufficient to prove this result.

# Nov 20

## Weyl-Kac formula for characters

Today we will prove Theorem 184 (Weyl-Kac formula), which is a formula for the character of an integrable $L(\widehat{\lambda})$.

Let's recall the setup. In fact, this all works for any Kac-Moody Lie algebra, but for us we'll focus on $\widehat{\mathfrak{g}}$, an affine Kac-Moody Lie algebra. In fact we need to consider the extended affine Kac-Moody Lie algebra $\widetilde{\mathfrak{g}}$, with Cartan decomposition

$$\widetilde{\mathfrak{g}} = \underbrace{\widehat{\mathfrak{n}}_-}_{\mathfrak{n}_-\oplus z^{-1}\cdot\mathfrak{g}[z^{-1}]} \oplus \underbrace{\widetilde{\mathfrak{h}}}_{\mathfrak{h}\oplus\mathbb{C}c\oplus\mathbb{C}d} \oplus \underbrace{\widehat{\mathfrak{n}}_+}_{\mathfrak{n}_+\oplus z\cdot\mathfrak{g}[z]}.$$

We also have an invariant symmetric form $\langle,\rangle$ which pairs $z^n\mathfrak{g}$ with $z^{-n}\mathfrak{g}$, after fixing some invariant symmetric form on $\mathfrak{g}$. We also need to specify what it does on $c, d$:

$$\begin{gathered}\langle c,d\rangle = 1,\\ \langle c, z^n\mathfrak{g}\rangle = 0,\\ \langle d, z^n\mathfrak{g}\rangle = 0.\end{gathered}$$

In particular, the restriction of this form to $\widetilde{\mathfrak{h}}$ is non-degenerate, hence we obtain an identification

$$\widetilde{\mathfrak{h}} \cong \widetilde{\mathfrak{h}}^*.$$

This allows us to talk about roots, weights, etc.

Now, we need some basis in $\widetilde{\mathfrak{g}}$. For any positive root $\alpha \in \Phi_+$, we have root spaces $\widehat{\mathfrak{g}}_\alpha \subset \widetilde{\mathfrak{g}}$. However, unlike in the finite-dimensional case, these subspaces may not be one-dimensional.

**Remark 191.**

In the case of $\widetilde{\mathfrak{g}}$, the subspace $\widehat{\mathfrak{g}}_\alpha$ has dimension greater than 1 precisely when the projection to the usual Cartan is zero.

Let's pick some basis of $\widehat{\mathfrak{g}}_\alpha$, given by $e^1_\alpha, \ldots, e^{n_\alpha}_\alpha$. This has a dual basis (with respect to the inner form, which is just a basis of $\widehat{\mathfrak{g}}_{-\alpha}$) given by $e^1_{-\alpha}, \ldots, e^{n_\alpha}_{-\alpha}$.

To have a basis for all of $\widetilde{\mathfrak{g}}$, we need a basis of $\widetilde{\mathfrak{h}}$. First we start with the simple roots $\alpha_i \in \widehat{\Pi}$. For the usual simple roots coming from $\mathfrak{g}$, we take the corresponding $h_{\alpha_i} \in \mathfrak{h} \subset \widetilde{\mathfrak{h}}$. The last simple root $\alpha_0$ coming from the highest root $\theta$ of $\mathfrak{g}$, we define $h_{\alpha_0} := c - h_\theta$. We also have the dual basis $\omega_i$ (with respect to the invariant form), which are the fundamental weights.
Lastly, we have the element $d$, completing the basis.

## Casimir operator

Naively, it should be the sum over the basis and the dual basis: $\sum x_a x^a$. Unfortunately, there is a problem: this element's action on category-$\mathcal{O}$ representations is not well-defined. So we need something which is well-defined.

Recall that in $\mathfrak{sl}_2$, the Casimir is $C = ef + fe + \frac{1}{2}h^2$, but we can rewrite it as $2fe + \frac{1}{2}h^2 + h$. The former is simpler to write, but its action is not well-defined on category-$\mathcal{O}$. So we want to generalize the latter presentation.

**Definition 192** (Casimir)**.**

We define the **Casimir operator** for $\widetilde{\mathfrak{g}}$ to be

$$C := 2 \sum_{\alpha \in \Phi_+} \sum_{i=1}^{n_\alpha} e^i_{-\alpha} e^i_\alpha \quad + \quad \sum \omega_i \cdot h_{\alpha_i} \quad + \quad 2h_{\widehat{\rho}},$$

where $\widehat{\rho} = \sum \omega_i$ is the sum of the fundamental weights.

We can prove this is central by checking its commutator with the Chevalley generators. It's clear it commutes with anything in $\widetilde{\mathfrak{h}}$, so then it suffices to compute its commutator with $e_{\alpha_i}$ and $f_{\alpha_i}$.

**Example 193.**

For $\widetilde{\mathfrak{g}}$, we have already seen the Casimir element: it is related to the Sugawara element $S_0$ by

$$C = S_0 + 2(c + h^\vee)d.$$

For example, for $\mathfrak{g} = \mathfrak{sl}_2$, the Casimir element in $\widetilde{\mathfrak{g}}$ is $C = S_0 + 2(c+2)d$.

## Main ideas in proof

Let $\widehat{\lambda}$ be a weight, and $M(\widehat{\lambda})$ be a Verma module. This means that we have

$$\begin{aligned} \langle \widehat{\lambda}, h_{\alpha_i} \rangle &= \lambda_i, \\ \langle \widehat{\lambda}, c \rangle &= k, \\ \langle \widehat{\lambda}, d \rangle &= n. \end{aligned}$$

The Casimir preserves the highest weight space, hence acts the highest weight vector by a scalar, and since it commutes with everything and the Verma module is generated by the highest weight vector, this implies that the Casimir acts by a fixed scalar on the entire Verma module. This scalar depends only on $\widehat{\lambda}$. Let us denote $\lambda := \widehat{\lambda}|_{\mathfrak{h}}$. Then

$$C|_{M(\widehat{\lambda})} = \big(\langle \lambda, \lambda + 2\rho \rangle + 2(c + h^\vee)n\big)\mathrm{id}.$$

The first main idea behind proving Theorem 184 (Weyl-Kac formula) is that if we write

$$\operatorname{ch} L(\widehat{\lambda}) = \sum a_{\widehat{\mu}} \cdot \operatorname{ch} M(\widehat{\mu}),$$

then we know that the Casimir acts by the same scalar on all of the $M(\widehat{\mu})$ that appear in the formula, and this scalar is the same as the scalar on $L(\widehat{\lambda})$ (consequently, on $M(\widehat{\lambda})$). This is because in the Grothendieck group, we have

$$[L(\widehat{\lambda})] = \sum a_{\widehat{\mu}}[M(\widehat{\mu})],$$

and then we can project onto the summand in $K_0$ corresponding to the eigenvalue of the Casimir.

Unfortunately, this is not sufficient for writing the character, because there are still too many possibilities. But it's a start.

The second main idea is if $L(\widehat{\lambda})$ is **integrable**, then we have elements $\tilde{s}_{\alpha_i} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ corresponding to simple roots $\alpha_i$ and the corresponding $\mathfrak{sl}_2$ that it generates. These elements send

$$\tilde{s}_{\alpha_i} : L(\widehat{\lambda})_\nu \to L(\widehat{\lambda})_{s_{\alpha_i}(\nu)}.$$

This means that $\operatorname{ch} L(\widehat{\lambda})$ is $\widehat{W}$-invariant. From this, we recover Proposition 183:

$$\operatorname{ch} L(\widehat{\lambda}) = \frac{\sum_{\text{dominant } \widehat{\mu}} \sum_{w \in \widehat{W}} a_{\widehat{\mu}} (-1)^w t^{w \cdot \widehat{\mu}}}{\prod_{\alpha \in \widehat{\Phi}_+} (1 - t^{-\alpha})},$$

where the dot action is given by $w \cdot \widehat{\mu} = w(\widehat{\mu} + \widehat{\rho}) - \widehat{\rho}$ and $a_{\widehat{\lambda}} = 1$, and the product in the denominator is taken with multiplicity (see Remark 185). A further explanation is given in the proof to Proposition 183.

The third observation is that $a_{\widehat{\mu}} \neq 0$ only if $\widehat{\mu} < \widehat{\lambda}$, i.e. $\widehat{\lambda} - \widehat{\mu}$ is a nonnegative integer combination of simple roots $\alpha_i$.

Now for the key lemma.

**Lemma 194.**

If $\widehat{\mu} < \widehat{\lambda}$ and both are dominant, then $\langle \widehat{\lambda}, \widehat{\lambda} + 2\widehat{\rho} \rangle > \langle \widehat{\mu}, \widehat{\mu} + 2\widehat{\rho} \rangle$.

***Proof.***
We want to show that $\langle \widehat{\lambda} + \widehat{\rho}, \widehat{\lambda} + \widehat{\rho} \rangle > \langle \widehat{\mu} + \widehat{\rho}, \widehat{\mu} + \widehat{\rho} \rangle$ (since this is equivalent to the statement in the lemma, which is easily checked by expanding the terms). Rearranging, we get

$$\langle \widehat{\lambda} + \widehat{\rho}, \widehat{\lambda} + \widehat{\rho} \rangle - \langle \widehat{\mu} + \widehat{\rho}, \widehat{\mu} + \widehat{\rho} \rangle = \left\langle \underbrace{\widehat{\lambda} + \widehat{\mu} + 2\widehat{\rho}}_{\text{dominant, } \langle -, \alpha_i \rangle > 0}, \quad \underbrace{\widehat{\lambda} - \widehat{\mu}}_{= \sum n_i \alpha_i,\, n_i \geq 0} \right\rangle > 0$$

since the right side of the inner product is a nonnegative integer combination of simple roots, and the left side of the inner product is dominant.

□

**Corollary 195.**

In Proposition 183, the coefficients $a_{\widehat{\mu}}$ for dominant $\widehat{\mu}$ are all zero whenever $\widehat{\mu} \neq \widehat{\lambda}$.

This proves Theorem 184 (Weyl-Kac formula), which we'll state again for convenience.

**Theorem 196** (Weyl-Kac formula)**.**

When $L(\widehat{\lambda})$ is integrable, then

$$\operatorname{ch} L(\widehat{\lambda}) = \frac{\sum_{w\in\widehat{W}}(-1)^w t^{w\cdot\widehat{\lambda}}}{\prod_{\alpha\in\widehat{\Phi}_+}(1-t^{-\alpha})},$$

where the product in the denominator is counted with multiplicity.

**Corollary 197.**

We also obtain a formula for the denominator:

$$\prod_{\alpha\in\widehat{\Phi}_+}(1-t^{-\alpha}) = \sum_{w\in\widehat{W}}(-1)^w\, t^{w\cdot 0}.$$

**Example 198.**

Let $\mathfrak{g}=\mathfrak{sl}_2$. The roots of $\widehat{\mathfrak{sl}_2}$ are

$$\widehat{\Phi} = \begin{cases}(2,0,n) & n\in\mathbb{Z},\\ (-2,0,n) & n\in\mathbb{Z},\\ (0,0,n) & 1\neq n\in\mathbb{Z}.\end{cases}$$

The positive roots are

$$\widehat{\Phi}_+ = \begin{cases}(2,0,n) & n\geq 0,\\ (-2,0,n) & n>0,\\ (0,0,n) & n>0.\end{cases}$$

Let's denote by $t$ the variable corresponding to the first coordinate and $q$ the variable corresponding to the third coordinate. Then

$$\prod_{\alpha\in\widehat{\Phi}_+}(1-t^{-\alpha}) = \prod_{n=1}^{\infty}(1-t^2q^{n-1})(1-t^{-2}q^n)(1-q^n) = \sum_m(-1)^m t^{2m} q^{\frac{m(m+1)}{2}}$$

by Theorem 55 (Jacobi triple product identity).

Next time, we'll even see how to derive the Jacobi triple product identity from Corollary 192.

# Nov 22

# Characters of integrable $\widehat{\mathfrak{sl}_2}$-modules

The goal of today is to relate these characters (which we computed last time, see Theorem 191 (Weyl-Kac formula)) to special and classical functions, especially theta functions.

**Setup for $\widehat{\mathfrak{sl}_2}$:**
We have the Cartan $\widetilde{\mathfrak{g}} \supset \widetilde{\mathfrak{h}} \cong \widetilde{\mathfrak{h}}^* = \mathrm{Span}\{\alpha, c, d\}$ (isomorphism induced by nondegenerate invariant scalar product). We have $\langle \alpha, \alpha \rangle = 2$ and $\langle c, d \rangle = 1$.
The simple roots are $\alpha_0 = c - \alpha$ and $\alpha_1 = \alpha$. The fundamental weights are $\omega_0 = d$ and $\omega_1 = \frac{1}{2}\alpha + d$. Therefore $\widehat{\rho} = 2d + \frac{1}{2}\alpha$.
Each of these simple roots defines a simple reflection. Let $\widehat{\mu} = kd + \frac{1}{2}\ell\alpha + nc$ be a generic element; if we consider $\widehat{\mu}$ as a weight, then $k$ represents the level, $\ell$ the highest weight of $\mathfrak{sl}_2$. Then

$$\begin{aligned} s_1 := s_{\alpha_1} : \alpha &\mapsto -\alpha, \\ c &\mapsto c, \\ d &\mapsto d, \end{aligned}$$

$$\begin{aligned} s_0 := s_{\alpha_0} : \alpha &\mapsto -\alpha + c, \\ c &\mapsto c, \\ d &\mapsto d + \alpha - c. \end{aligned}$$

Together, $s_{\alpha_1}$ and $s_{\alpha_0}$ freely generate the affine Weyl group $\widehat{W}$. In particular, this means that $\widehat{W} = \langle s_0, s_1 \mid (s_0)^2 = (s_1)^2 = 1 \rangle$, and so any even element is just $(s_1 s_0)^r$ for some $r$, while any odd element is just $(s_1 s_0)^r s_1$ for some $r$.

We can view this as an affine reflection group of the affine line $\mathbb{A}^1$, with one reflection about $0$ and another reflection about $1$. Then $s_1 s_0$ is just a shift. Let's make the convention

$$t_r := (s_1 s_0)^r.$$

Then by induction, we can compute the action of $t_r$:

$$\begin{aligned} t_r : \alpha &\mapsto \alpha + 2rc, \\ c &\mapsto c, \\ d &\mapsto d - r\alpha - r^2 c. \end{aligned}$$

# Character formula

One way to understand the character is by trace:

$$\mathrm{ch}_L = \mathrm{tr}_L \, \exp(h) : \widetilde{\mathfrak{h}} \to \mathbb{C}.$$

To relate it with classical analysis, we should renormalize $\widetilde{\mathfrak{h}} \ni h = 2\pi i \left(\frac{1}{2} z\alpha - \tau d + uc\right)$. Then Theorem 191 (Weyl-Kac formula) becomes

$$\mathrm{ch}_{L(\widehat{\lambda})}(z,\tau,u) = \frac{\sum_{w\in\widehat{W}} \exp(\langle w(\widehat{\lambda}+\widehat{\rho}) - \widehat{\rho}, h\rangle)\cdot(-1)^w}{\prod_{\beta\in\Phi_+}(1-\exp(-\beta_i h))}.$$

**Numerator**

Let's understand the numerator as a function of $z, \tau, u$. Write $\widehat{\mu} := \widehat{\lambda} + \widehat{\rho} = \frac{1}{2}\ell\alpha + kd + nc$. When $(-1)^w = 1$, then $w = t_r$ for some $r \in \mathbb{Z}$, so the numerator (ignoring the factor of $\langle -\widehat{\rho}, h\rangle$) is

$$\exp\langle t_r(\widehat{\mu}) - \widehat{\rho}, h\rangle = \exp 2\pi i\left(\left(\frac{1}{2}\ell - kr\right)z - \left(n + \frac{1}{2}r\ell - r^2 k\right)\tau + ku\right).$$

If we replace $r$ by $\frac{\ell}{2k} - r$ we get

$$\sum_{r\in\frac{\ell}{2k}+\mathbb{Z}} \exp 2\pi i\left(krz + \left(kr^2 - \left(n + \frac{\ell^2}{4k}\right)\right)\tau + ku\right).$$

When $(-1)^w = -1$ we get

$$-\sum_{r\in-\frac{\ell}{2k}+\mathbb{Z}} \exp 2\pi i\left(krz + \left(kr^2 - \left(n + \frac{\ell^2}{2k}\right)\right)\tau + ku\right).$$

The key point is that these are *almost* Jacobi-Riemann theta functions.

**Definition 199** (theta function)**.**

We define the **theta function** to be

$$\Theta_{\ell,k}(\tau,z,u) := \exp(2\pi i\, ku)\sum_{r\in\frac{\ell}{2k}+\mathbb{Z}} \exp\left(2\pi i\, k(r^2\tau + rz)\right).$$

This means that the **numerator** of $\mathrm{ch}_{L(\widehat{\lambda})}$ is (up to some factor)

$$(\Theta_{\ell+1,k+2} - \Theta_{-\ell-1,k+2})q^{n+\frac{(\ell+1)^2}{4(k+2)}}\exp\langle -\!\!-\widehat{\rho}, h\rangle, \qquad q = \exp(2\pi i\tau).$$

The **denominator** is

$$(\Theta_{1,2} - \Theta_{-1,2})q^{\frac{1}{8}}\exp\langle -\widehat{\rho}, h\rangle.$$

**Theorem 200** (character of integrable $\mathfrak{sl}_2$-module)**.**

The character of integrable $L(\widehat{\lambda})$ for $\widehat{\mathfrak{sl}_2}$ where $\lambda = kd + \frac{1}{2}\alpha + nc$ is given by

$$\mathrm{ch}_{L(\widehat{\lambda})}(z,\tau,u) = q^{s_{\widehat{\lambda}}} \frac{\Theta_{\ell+1,k+2} - \Theta_{-\ell-1,k+2}}{\Theta_{1,2} - \Theta_{-1,2}},$$
$$s_{\widehat{\lambda}} = \frac{(\ell+1)^2}{4(k+2)} + n - \frac{1}{8}.$$

For $\widehat{\lambda} = 0$, we actually get the Jacobi triple product identity.

Our next goal is to compute $\mathrm{ch}_{L(0,1)}$, corresponding to $\widehat{\lambda} = d$.

## Computing $\mathrm{ch}_{L(0,1)}$

According to Theorem 195 (character of integrable $\mathfrak{sl}_2$-module), we have

$$\mathrm{ch}_{L(0,1)} = q^{\frac{1}{24}} \frac{\Theta_{1,3} - \Theta_{-1,3}}{\Theta_{1,2} - \Theta_{-1,2}}.$$

In fact, it has a much simpler form:

**Proposition 201** (character of $L(0,1)$)**.**

The character of the $\widehat{\mathfrak{sl}_2}$-module $L(0,1)$ is

$$\mathrm{ch}_{L(0,1)} = \frac{\Theta_{0,1}}{\varphi(q)}, \qquad \varphi(q) = \prod_{n=1}^{\infty}(1-q^n).$$

In fact there are several ways of proving this. One, which we may discuss later, is by constructing $L(0,1)$ as a sum of Fock spaces, and using that the character of a Fock space is $\frac{1}{\varphi(q)}$. The way we'll see now is via the Kac-Peterson multiplication rule for theta functions. In fact the proof is a fairly straightforward, if lengthy, computation, so we only give a very brief sketch.

**Theorem 202** (Kac-Peterson, multiplication rule for theta functions)**.**

$$\Theta_{\ell,k} \cdot \Theta_{\ell',k'} = \sum_{j \in \mathbb{Z}/(k+k')\mathbb{Z}} \psi_j^{(k,k',\ell,\ell')}(q) \cdot \Theta_{\ell+\ell'+2kj,k+k'},$$
$$\psi_j^{(k,k',\ell,\ell')}(q) = \sum_r q^{kk'(k+k')r^2}.$$

The idea of the proof is to write the product as

$$\Theta \cdot \Theta = \sum_{\substack{r,r' \\ r=\frac{\ell}{2k}+i,\, i\in\mathbb{Z} \\ r'=\frac{\ell'}{2k'}+i',\, i'\in\mathbb{Z}}} q^{kr^2+k'r'^2} \exp 2\pi(kr+k'r')z,$$

and we can even assume $u=0$ (as nothing really depends on $u$). Then by making the clever change of variables

$$\begin{aligned} s: \quad (k+k')s = r-r' &\in \frac{\ell k' - \ell' k + 2kk' \cdot \overbrace{(i-i')}^{j}}{2kk'(k+k')} + \mathbb{Z}, \\ s': \quad (k+k')s' = kr - k'r' &\in \frac{\ell+\ell'+2kj}{2(k+k')} + \mathbb{Z}. \end{aligned}$$

Now there is a bijection between pairs $(r, r')$ and triples $(s, s', j)$; changing the variables, we get the result.

Now for the sketch of the proof of Proposition 196 (character of $L(0,1)$). We'll complete this next time, but we basically apply Theorem 197 (Kac-Peterson, multiplication rule for theta functions) and Theorem 57 (Euler pentagonal number theorem) to get

$$\Theta_{1,0}(\Theta_{1,2} - \Theta_{-1,2}) = q^{\frac{1}{24}}\varphi(q)(\Theta_{1,3} - \Theta_{-1,3}).$$

# Nov 27

## Integrable modules over $\widehat{\mathfrak{sl}_2}$; Virasoro algebra

Let's first consider more generally some finite-dimensional simple $\mathfrak{g}$ (not necessarily $\mathfrak{sl}_2$), and its corresponding affine Kac-Moody Lie algebra $\widehat{\mathfrak{g}}$. Denote by $\mathcal{O}_{\text{int}}(\widetilde{\mathfrak{g}})$ the category of integrable category-$\mathcal{O}(\widetilde{\mathfrak{g}})$ modules. We have already classified all of the simple objects in this category in Proposition 181. Therefore, to understand the category $\mathcal{O}_{\text{int}}(\widetilde{\mathfrak{g}})$ completely, it remains to understand how to build these modules from simple ones.

**Theorem 203.**

Category $\mathcal{O}_{\text{int}}(\widetilde{\mathfrak{g}})$ is semisimple, and simple objects are $L(\widehat{\lambda})$ with $\widehat{\lambda}$ dominant integral (i.e., for any simple coroot $\alpha_i^\vee$, we have $\langle \widehat{\lambda}, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}$; see Proposition 181).

***Proof.***
The statement of simple objects was already proven in Proposition 181.

First, we explicitly describe $L(\widehat{\lambda})$.

**Lemma 204.**

When $L(\widehat{\lambda})$ is integrable, then it is the quotient of the Verma module $M(\widehat{\lambda})$ by the sum over its Verma submodules $M(s_i \cdot \widehat{\lambda})$ generated by simple reflections $s_i$:

$$L(\widehat{\lambda}) = M(\widehat{\lambda}) \Big/ \sum_i M(s_i \cdot \widehat{\lambda}).$$

***Proof.***

First, the surjection from the quotient onto $L(\widehat{\lambda})$ is clear: we already know that the $M(s_i \cdot \widehat{\lambda})$ are submodules, so they must vanish in the surjection from $M(\widehat{\lambda})$. It remains to show that this quotient is simple. First, note that the quotient is integrable, as it is integrable with respect to every simple root $\mathfrak{sl}_2$, as we mod out by every Verma module corresponding to a simple reflection. It is also a highest-weight module. Suppose that this quotient were not simple; then there is a singular vector $v_{\widehat{\mu}}$ of weight $\widehat{\mu}$. Now we know that $\widehat{\lambda} - \widehat{\mu}$ is a nonnegative integer combination of simple roots, i.e. $\widehat{\mu} < \widehat{\lambda}$; on the other hand, since the quotient is a highest-weight module, and this $\widehat{\mu}$ is a singular vector generating a submodule, $\widehat{\mu}$ must be dominant integrable. But then Lemma 189 tells us that this is impossible: the Casimir operator will have different eigenvalues on these two submodules.

□

Now let $V \in \mathcal{O}_{\text{int}}(\widetilde{\mathfrak{g}})$ be any integrable object, and let $V^{\text{sing}} := V^{\widehat{\mathfrak{n}}_+}$ be the subspace of singular vectors. Then

$$V^{\text{sing}} = V^{\widehat{\mathfrak{n}}_+} = \bigoplus_{\widehat{\mu}} V_{\widehat{\mu}}^{\widehat{\mathfrak{n}}_+}$$

is the direct sum of its weight components. For every $\widehat{\mu}$ in this decomposition, a singular vector induces a map $M(\widehat{\mu}) \to V$ factoring through $L(\widehat{\mu})$, the unique integrable quotient. This means that the submodule in $V$ generated by $V^{\text{sing}}$ is actually just the direct sum of the integrable quotients (counted with multiplicity):

$$V \supset V' := \mathcal{U}(\widetilde{\mathfrak{g}}) V^{\text{sing}} = \bigoplus_{\widehat{\mu}} L(\widehat{\mu}).$$

It remains to show that this is in fact all of $V$. A priori, this might be false, and $V/V' \neq 0$. So suppose for the sake of contradiction that $V/V' \neq 0$. Take a highest weight vector $v_{\widehat{\mu}} \in V/V'$ and lift it to $\widetilde{v_{\widehat{\mu}}} \in V$. Then $\widehat{\mathfrak{n}}_+ \widetilde{v_{\widehat{\mu}}} \subset V'$. So there is $x \in \mathcal{U}(\widehat{\mathfrak{n}}_+)$ such that $x \cdot \widetilde{v_{\widehat{\mu}}} = v_{\widehat{\gamma}} \in V'$ is a highest weight vector. Now observe that

$$\widehat{\gamma} = \widehat{\mu} + \sum n_i \alpha_i, \qquad n_i \in \mathbb{Z}_{\geq 0}.$$

Moreover, the eigenvalue of Casimir on $\widetilde{v_{\widehat{\mu}}}$ and $v_{\widehat{\gamma}}$ are the same. Also, both $\widehat{\mu}$ and $\widehat{\gamma}$ are dominant. So we arrive at the same scenario: Lemma 189 tells us this is impossible, so $V/V' = 0$ and $V = V' = \bigoplus_{\widehat{\mu}} L(\widehat{\mu})$ (counted with multiplicity).

□

So this means that we have a very nice subcategory $\mathcal{O}_{\mathrm{int}} \subset \mathcal{O}$. However, despite the nice property of being semisimple, $\mathcal{O}_{\mathrm{int}}$ has a rather complicated structure: the tensor product rule (for $L(\widehat{\lambda})$) is rather complicated.

In order to decompose the tensor products of the irreducible integrable modules even for the case of $\mathfrak{g} = \mathfrak{sl}_2$, we need some recollections of important identities.

**Recollection.** Recall Definition 194 (theta function):

$$\Theta_{\ell,k}(\tau, z, u) = \exp(2\pi i\, ku) \cdot \sum_{r \in \frac{\ell}{2k} + \mathbb{Z}} \exp\left(2\pi i\, k(r^2\tau + rz)\right).$$

Also recall Theorem 195 (character of integrable $\mathfrak{sl}_2$-module), which states that for $\widehat{\lambda} = kd + \frac{1}{2}\ell\alpha + nc$, then

$$\mathrm{ch}_{L(\widehat{\lambda})}(z, \tau, u) = q^{s_{\widehat{\lambda}}} \frac{\Theta_{\ell+1,k+2} - \Theta_{-\ell-1,k+2}}{\Theta_{1,2} - \Theta_{-1,2}},$$
$$s_{\widehat{\lambda}} = \frac{(\ell+1)^2}{4(k+2)} + n - \frac{1}{8}.$$

The basic idea (which works for arbitrary $\mathfrak{g}$, but we work with $\mathfrak{sl}_2$ here to make use of the concrete connection to theta functions) is to just multiply the characters and attempt to write it out as a linear combination of characters of irreducible integrable modules (since the tensor product is clearly still integrable, so by semisimplicity it decomposes into a direct sum of irreducible integrable modules). Naturally, we'll need Theorem 197 (Kac-Peterson, multiplication rule for theta functions):

$$\Theta_{\ell,k} \cdot \Theta_{\ell',k'} = \sum_{j \in \mathbb{Z}/(k+k')\mathbb{Z}} \psi_j^{(k,k',\ell,\ell')}(q) \cdot \Theta_{\ell+\ell'+2kj,k+k'},$$
$$\psi_j^{(k,k',\ell,\ell')}(q) = \sum_r q^{kk'(k+k')r^2}.$$

**Remark 205.**

There may be a minor mistake in the formulas here.

Now once we have this formula, we can deduce Proposition 196 (character of $L(0,1)$), which we'll restate here for convenience.

**Proposition 206** (character of $L(0,1)$).

The character of the $\widehat{\mathfrak{sl}_2}$-module $L(0,1)$ is

$$\mathrm{ch}_{L(0,1)} = \frac{\Theta_{0,1}}{\varphi(q)}, \qquad \varphi(q) = \prod_{n=1}^{\infty}(1-q^n).$$

(This is a copy of Proposition 196 (character of $L(0,1)$).)

***Proof.***
We need to prove that

$$\Theta_{0,1}\cdot(\Theta_{1,2}-\Theta_{-1,2}) = q^{\frac{1}{24}}\varphi(q)(\Theta_{1,3}-\Theta_{-1,3}).$$

So we will use Theorem 197 (Kac-Peterson, multiplication rule for theta functions) to expand the left-hand side:

$$\begin{aligned}
\Theta_{0,1}\cdot(\theta_{1,2}-\Theta_{-1,2}) &= (\Theta_{1,3}-\Theta_{-1,3})\cdot\left(\sum_{r\in-\frac{1}{12}+\mathbb{Z}} q^{6r^2} - \sum_{r'\in\frac{5}{12}+\mathbb{Z}} q^{6r'^2}\right),\\
&= (\Theta_{1,3}-\Theta_{-1,3})\cdot\left(q^{\frac{1}{24}}\sum_{j\in\mathbb{Z}}(-1)^j q^{\frac{3j^2+j}{2}}\right),\\
&= (\Theta_{1,3}-\Theta_{-1,3})\cdot q^{\frac{1}{24}}\cdot\varphi(q),
\end{aligned}$$

by Theorem 55 (Jacobi triple product identity).

□

**Corollary 207.**

We can describe $L(0,1)$ as the sum of Fock spaces.

$$L(0,1) = \bigoplus_{r\in\mathbb{Z}} F_r,$$

where $F_r$ is the Fock space for the copy of the Heisenberg algebra $\{h[n] \mid n\in\mathbb{Z}\}\subset\widetilde{\mathfrak{sl}_2}$.

More concretely, $F_r$ is generated by $v_r$ where $h[n]v_r = 0$ for $n>0$, $h[0]v_r = 2r\,v_r$, and $dv_r = -r^2\,v_r$.

**Example 208.**

Let $\mathfrak{g}=\mathfrak{sl}_2$. Let's try to decompose $L(0,1)\otimes L(\ell,k)$. It turns out that:

**Theorem 209.**

$$L(0,1) \otimes L(\ell,k) = \bigoplus_{r \in I} \underbrace{\mathbb{V}(k,\ell;r)}_{\text{some multiplicity space}} \otimes L(\ell - 2r, k+1),$$

$$I = \left\{ r \in \mathbb{Z} \mid -\frac{1}{2}(k+1-\ell) \le r \le \frac{1}{2}\ell \right\}.$$

The $\mathbb{V}$ is an infinite-dimensional space, but it is graded with respect to $d$ (and has finite-dimensional graded components). It is possible to write the character of $\mathbb{V}$ with respect to this grading, so it will be a power series in $q$. It is very interesting; it even carries an action by $Vir$. We will discuss this next time!

# Nov 29

Today's main focus will be:

## Goddard-Kent-Olive construction of representations of Virasoro

But first, let's finish what we were discussing from last class.

### Tensor product rule for $\widehat{\mathfrak{sl}_2}$

It is possible to write the product of the characters of integrable modules as a linear combination of the characters of integrable modules:

$$\mathrm{ch}_{L(0,1)} \cdot \mathrm{ch}_{L(\ell,k)} = \sum_r \psi_{k,\ell;r} \cdot \mathrm{ch}_{L(\ell-2r,k+1)}$$

where the coefficients are power series in $q = \exp(\tau)$.

**Proposition 210.**

Let $r \in \mathbb{Z}$ such that $-\frac{1}{2}(k+1-\ell) \le r \le \frac{1}{2}\ell$. Then

$$\psi_{k,\ell;r} = \frac{1}{\varphi(q)} \sum_{j \in \mathbb{Z}} q^{(k+1)(k+3)j^2 + ((\ell+2)+2(\ell+1-r)(k+2))j + (\ell+1-r)^2}.$$

***Proof.***
The proof is found in Kac-Raina "Bombay lectures" and uses Theorem 197 (Kac-Peterson, multiplication rule for theta functions).

First we have

$$\Theta_{\ell,k}\Theta_{\ell',k'} = \sum_j \underbrace{d_j^{k,k',\ell,\ell'}(q)}_{=\sum_r q^{kk'(k+k')r^2},\quad r\in\mathbb{Z}+\frac{k'\ell-k\ell'+2jkk'}{2kk'(k+k')}} \Theta_{\ell+\ell'+2kj,\, k+k'}$$

$$\implies \mathrm{ch}_{L(0,1)} = q^{\frac{1}{24}}\frac{\Theta_{1,3}-\Theta_{-1,3}}{\Theta_{1,2}-\Theta_{-1,2}} = \frac{\Theta_{0,1}}{\varphi(d)}.$$

This is a good thing for multiplication in the Weyl-Kac formula, because $\frac{\Theta_{0,1}}{\varphi(d)}$ is easy to multiply by $q^{s_{\hat\lambda}}\frac{\Theta_{\ell+1,k}-\Theta_{-\ell-1,k}}{\Theta_{1,2}-\Theta_{-1,2}}$. Then we just expand the product of characters using Theorem 195 (character of integrable $\mathfrak{sl}_2$-module) and Theorem 197 (Kac-Peterson, multiplication rule for theta functions) and after a lot of work, we get the result.

□

**Goal:** explain that $\psi$ is a character of some Virasoro module.

We know from Nov 27 that the tensor product $L(0,1)\otimes L(\ell,k)$ decomposes into a direct sum of irreducible integrable $\widehat{\mathfrak{sl}_2}$-modules, with some multiplicity spaces $M_r$.

$$L(0,1)\otimes L(\ell,k) = \bigoplus_r \underbrace{M_r}_{\substack{\circlearrowleft \\ Vir}} \otimes L(\ell-2r, k+1).$$

It will turn out that the multiplicity spaces $M_r$ carry a $Vir$-action.

### Example 211.

**Baby example:** for $\mathfrak{sl}_2$-modules, we have

$$L(\lambda_1)\otimes L(\lambda_2) = L(\lambda_1+\lambda_2)\oplus L(\lambda_1+\lambda_2-2)\oplus\cdots+\oplus L(\lambda_1-\lambda_2)$$

where $\lambda_1>\lambda_2$. This decomposition can be obtained as the decomposition with respect to some operator, namely the **diagonal Casimir** $\Delta(C)\in\Delta(\mathcal{U}(\mathfrak{sl}_2))\subset\mathcal{U}(\mathfrak{sl}_2)\otimes\mathcal{U}(\mathfrak{sl}_2)$, given by the Casimir the diagonal of $\mathfrak{sl}_2$ inside the tensor product. Each summand in the decomposition is uniquely determined by the eigenvalue of this diagonal Casimir, because $C\Big|_{L(\lambda)} = \frac{\lambda(\lambda+2)}{2}$.

In fact we can even describe the algebra of all operators commuting with the diagonal $\mathfrak{sl}_2$ inside this algebra:

$$(\mathcal{U}(\mathfrak{sl}_2)\otimes\mathcal{U}(\mathfrak{sl}_2))^{\Delta(\mathfrak{sl}_2)} = \mathbb{C}[C^{(1)},C^{(2)},\Delta(C)] = \mathbb{C}[C^{(1)},C^{(2)},\Omega],$$

$$\Delta(C) := \Delta(e)\Delta(1)+\cdots = C^{(1)}+C^{(2)}+2\left(\underbrace{e\otimes f+f\otimes 3+\frac{1}{2}h\otimes h}_{\Omega}\right),$$

where $C^{(1)}$ is the Casimir in the first copy, $C^{(2)}$ is the Casimir in the second copy, and $\Delta(C)$ is the diagonal Casimir (which we related to $C^{(1)}$ and $C^{(2)}$ above).

We want to do something similar in our case, with affine $\widehat{\mathfrak{sl}_2}$.

Copying this strategy to our case, $M_r$ is naturally a module over the centralizer subalgebra $\left(\mathcal{U}(\widehat{\mathfrak{sl}_2}) \otimes \mathcal{U}(\widehat{\mathfrak{sl}_2})\right)^{\Delta(\widehat{\mathfrak{sl}_2})}$, and even $\left(\mathcal{U}(\widehat{\mathfrak{sl}_2})_1 \widetilde{\otimes} \mathcal{U}(\widehat{\mathfrak{sl}_2})_k\right)^{\Delta(\widehat{\mathfrak{sl}_2})}$ (because we specified the levels $1$ and $k$, and since we're in category $\mathcal{O}$ we can take the local completion, as generally speaking, there may not be anything in the strict centralizer if we don't take completion).

**Idea:** do the same thing we just did in the baby example, but for Sugawara.

**Recollection on Sugawara elements**
We have $\mathfrak{g} \ni x_a, x^a$ - dual bases with respect to $\langle , \rangle$. Recall that

$$S(u) = \quad : \sum_a x_a(u) x^a(u) : \quad = \sum S_i u^{-i-2}.$$

Then $[S_i, x[r]] = (\text{constant depending on } r) \cdot x[r+1] \pmod{c = k}$ for $x \in \mathfrak{g}$. On the other hand, $[S_i, -]$ is a derivation, and we already classified all of the derivations of the affine Lie algebras: there are inner derivations, plus the Witt algebra. So this $[S_i, -] = ? \cdot z^{n+1}\partial_z$ for some constant, and this constant is easy to compute (for example we can check with $x[1]$ for some $x \in \mathfrak{g}$); what we get is that $[S_i, -] = 2(c + h^\vee) z^{i+1} \partial_z$ where $h^\vee$ is the Coxeter number. For $\mathfrak{sl}_2$ we get $2(c+2)$.

Now define $L_i := -\frac{1}{2(c+2)} S_i$ and this generates a copy of Virasoro. The claim is that the commutator between these guys is what we get from commutating the Witt algebra, plus something which commutes with everything, giving us Virasoro.

**Compute the central extension for $\mathfrak{sl}_2$**
We'll also see how to generalize it.

Recall that in Virasoro, $[L_i, L_{-i}] = 2iL_0 + \frac{i^3 - i}{12} c$. This central extension is determined by a cocycle, and there is a one-dimensional space of cocycles modulo coboundaries (so this cocycle is unique up to scaling, after specifying that it is $0$ on the input $i = 1$). To determine the constant that $c$ acts by, we compute the action of $[L_2, L_{-2}]$ on any representation. For us, we want to compute the constant that $c$ acts by after identifying the Virasoro algebra with the space of (modified) Sugawara elements. To compute the constant, we compute its action on the highest vector (the vacuum vector $v_0$) of $\mathbb{V}(0, k) = \mathcal{U}(\widehat{\mathfrak{g}}) \otimes_{\mathcal{U}(\mathfrak{g}[z] + \mathbb{C}c)} \mathbb{C} = \mathcal{U}(z^{-1}\mathfrak{g}[z^{-1}])v_0$, where $\mathfrak{g}[z]$ acts by $0$ on $\mathbb{C}$, and $c$ acts by $k$. (So this is a bit smaller than the Verma module; it's a "Verma module" with highest weight $0$ and level $k$, modulo $\widehat{\mathfrak{n}}_-$.)

We now note that $L_{>0} v_0 = 0$, and moreover $L_0 v_0 = 0$ as well, because all terms except the Casimir of the finite $\mathfrak{g}$ vanish, but this acts by zero as well because it's the trivial representation $\mathbb{C}$. We also see that $L_{-1} v_0 = (\sum x_a[-1] x^a[0]) v_0 = 0$, since $x^a[0] v_0 = 0$. This means that $[L_1, L_{-1}] = 2L_0 + 0 = 2L_0$, so in particular the cocycle is uniquely determined now by the commutator $[L_2, L_{-2}]$. We find that

$$[L_2, L_{-2}]v_0 = L_2\left(-\frac{1}{2(k+2)}\left(e[-1]f[-1] + f[-1]e[-1] + \frac{1}{2}h[-1]^2\right)\right)v_0,$$
$$= \left(\sum x_a[1]x^a[-1]\right)v_0.$$

It remains to use the following lemma.

**Lemma 212.**

$\sum_a [x_a[1], x^a[-1]] = (\dim \mathfrak{g}) \cdot c.$

***Proof.***
This is a $\mathfrak{g}$-invariant degree $0$ element, so it is proportional to $c$. On the other hand, each term is just $c$, and the $x_a$ form a basis, so there's exactly $\dim \mathfrak{g}$ of them.

□

Finally we arrive at the central charge:

**Proposition 213.**

The central charge of this copy of Virasoro is $c = \frac{(\dim \mathfrak{g})\cdot k}{k+h^\vee}$.

***Proof.*** This follows from $\frac{2^3-2}{12}c = \frac{1}{2}\frac{(\dim \mathfrak{g})k}{k+h^\vee}$. □

We've found the central charge, but there's more work to do! We still need to carry out the strategy above, with the diagonal Casimir.

For next time: In $\mathcal{U}(\widehat{\mathfrak{sl}_2}) \otimes \mathcal{U}(\widehat{\mathfrak{sl}_2}) \supset \Delta(\widehat{\mathfrak{sl}_2})$. Then we have $\Delta(L_i) - L_i^{(1)} - L_i^{(2)}$ commuting with everything in $\Delta(\widehat{\mathfrak{sl}_2})$. We will see that in fact these elements satisfy the Virasoro relations, so from the general theory, we get a Virasoro-module structure on the multiplicity spaces $M_r$. Then when we substitute the level $1$ on the first tensor and generic on the second, then we get generic Verma modules for Virasoro for the multiplicity spaces. And when we consider smaller modules over the second tensor, then we get smaller multiplicity spaces, and this will help us to study the representation theory of these objects.

# Dec 1

## Goddard-Kent-Olive construction of Virasoro reps

First: the Sugawara construction.

Let $\mathfrak{g}$ be a simple Lie algebra. We have an embedding $Vir \hookrightarrow \widetilde{\mathcal{U}(\widehat{\mathfrak{g}})_k}$ for any $k \neq -h^\vee$.
Then

$$L^{\mathfrak{g}}(u) = \frac{1}{2(k+h^\vee)} \cdot \left( :\sum x_a(u) x^a(u): \right)$$

where $x_a(u), x^a(u)$ are dual bases of $\mathfrak{g}$ with respect to $\langle , \rangle$.
We can extend this construction to any reductive

$$\mathfrak{g} = \bigoplus_{s=1}^{N} \mathfrak{g}_s, \quad L^{\mathfrak{g}}(u) := \sum_{s=1}^{N} L^{\mathfrak{g}_s}(u).$$

The central charge of the Virasoro embedding can be computed as follows. For a simple Lie algebra $\mathfrak{g}$, we have

$$[L_i, L_j] = (i-j)L_{i+j} + \delta_{i+j=0} \frac{i^3 - i}{12} c, \quad c = \frac{\dim \mathfrak{g} \cdot k}{k + h^\vee}.$$

As an example, for $\mathfrak{sl}_2$, $h^\vee = 2$, so $c = \frac{3k}{k+2}$.
In general,

$$c = \sum_{s=1}^{N} c^{\mathfrak{g}_s} = \sum_{s=1}^{N} \frac{\dim \mathfrak{g}_s \cdot k_s}{k_s + h_s^\vee}.$$

(This includes abelian $1$-dimensional $\mathfrak{g}$, then $L(u) = \frac{1}{2} \quad : a(u)^2 :$, and $h^\vee = 0$.)

Now let $\mathfrak{g} \supset \mathfrak{p}$ a reductive subalgebra. Then in the completed enveloping algebra $\widetilde{\mathcal{U}(\widehat{\mathfrak{g}})}$ we have two copies of Virasoro: $L_i^{\mathfrak{g}}$ and $L_i^{\mathfrak{p}}$. We know that

$$\begin{aligned} [L_i^{\mathfrak{g}}, x[n]] &= -n \cdot x[n+i] \quad \forall x \in \mathfrak{g}, \\ [L_i^{\mathfrak{p}}, y[n]] &= -n \cdot y[n+i] \quad \forall y \in \mathfrak{p}. \end{aligned}$$

This means that we can consider the differences

$$L_i^{\mathfrak{g}} - L_i^{\mathfrak{p}} =: L_i \in \widetilde{\mathcal{U}(\widehat{\mathfrak{g}})}_k^{\hat{\mathfrak{p}}}.$$

**Proposition 214.**

The $L_i$ satisfy the Virasoro relations

$$[L_i, L_j] = (i-j)L_{i+j} + \delta_{i+j=0} \frac{i^3 - i}{12} c, \quad c := c^{\mathfrak{g}} - c^{\mathfrak{p}}.$$

***Proof.***
We just compute:

$$
\begin{aligned}
[L_i, L_j] &= [L_i, L_j^{\mathfrak{g}} - L_j^{\mathfrak{p}}], \\
&= [L_i, L_j^{\mathfrak{g}}], \qquad\qquad L_i \text{ commutes with everything in } \widetilde{\mathcal{U}(\widehat{\mathfrak{g}})}^{\hat{\mathfrak{p}}} \\
&= [L_i^{\mathfrak{g}}, L_j^{\mathfrak{g}}] - [Li^{\mathfrak{p}}, L_j^{\mathfrak{g}}], \\
&= [L_i^{\mathfrak{g}}, L_j^{\mathfrak{g}}] - [L_i^{\mathfrak{p}}, L_j + L_j^{\mathfrak{p}}], \\
&= [L_i^{\mathfrak{g}}, L_j^{\mathfrak{g}}] - [L_i^{\mathfrak{p}}, L_j^{\mathfrak{p}}], \\
&= (i-j)(L_i^{\mathfrak{g}} - L_i^{\mathfrak{p}}) + \delta_{i+j=0} \frac{i^3 - i}{12}(c^{\mathfrak{g}} - c^{\mathfrak{p}}).
\end{aligned}
$$

□

**Corollary 215.**

Let $M \in \mathcal{O}(\widehat{\mathfrak{g}})_k$ and $N \in \mathcal{O}(\hat{\mathfrak{p}})_k$. Then $\mathrm{Hom}_{\hat{\mathfrak{p}}}(N, M)$ is a "category-$\mathcal{O}$", i.e. upper bounded with respect to the $d$-grading, module over $Vir$.

**Example 216.**

Let $\mathfrak{g} = \mathfrak{sl}_2$ and $\mathfrak{p} = \mathfrak{h}$. Take a Verma module with generic $\lambda$ for $\widehat{\mathfrak{sl}_2}$. Then

$$M(\lambda) = \bigoplus_{\nu} N_\nu \otimes \underbrace{F_\nu}_{\text{Fock space}}.$$

Roughly, $M(\lambda)$ is the tensor-cubed of a Fock space, i.e. it looks like $\mathbb{C}[e[n], h[n], f[n+1]]_{n<0}$. That means that $N_\nu$ should be something like the tensor-square of a Fock space, which is much larger than the Verma for a Virasoro!

## Main example

Let $\mathfrak{g} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, and let $\mathfrak{p} = \Delta(\mathfrak{sl}_2)$ be the diagonal.
Let's consider **specific levels**: we want (some arbitrary) $k$ for the first copy of $\mathfrak{sl}_2$ and $1$ on the second copy.

On the level $1$, we have the really nice module $L(0,1)$ which we already studied; it's integrable with highest weight $0$.

Then the general theory we constructed gives us the following. Take any irreducible module $L(\widehat{\lambda})$ of the level $k$. Then the tensor product $L(\widehat{\lambda}) \otimes L(0,1)$ has level $k+1$. Now consider the irreducible module with level $k+1$, $L(\widehat{\mu})$. Then the space

$$\mathrm{Hom}_{\widehat{\mathfrak{sl}_2}}(\underbrace{L(\widehat{\mu})}_{\text{level } k+1}, \underbrace{L(\widehat{\lambda})}_{\text{level } k} \otimes L(0,1))$$

is a module over $Vir$ with size smaller than or equal to that of the Verma module over Virasoro (at the very least, comparable in size).

**Remark 217.**

If we can understand this, then potentially this could give us constructions of many irreducible Virasoro modules, from relatively easy constructions. Of course we can define Verma modules for Virasoro, but it is a bit difficult to explore their irreducible quotients because we don't have the same nice structure as we have in the Kac-Moody case: we don't have Weyl groups, we don't have reflections, we don't have Casimir elements. But this is some construction which reduces at least some of the constructions for Virasoro to that of $\widehat{\mathfrak{sl}_2}$, which makes it very interesting.

How do we see this?

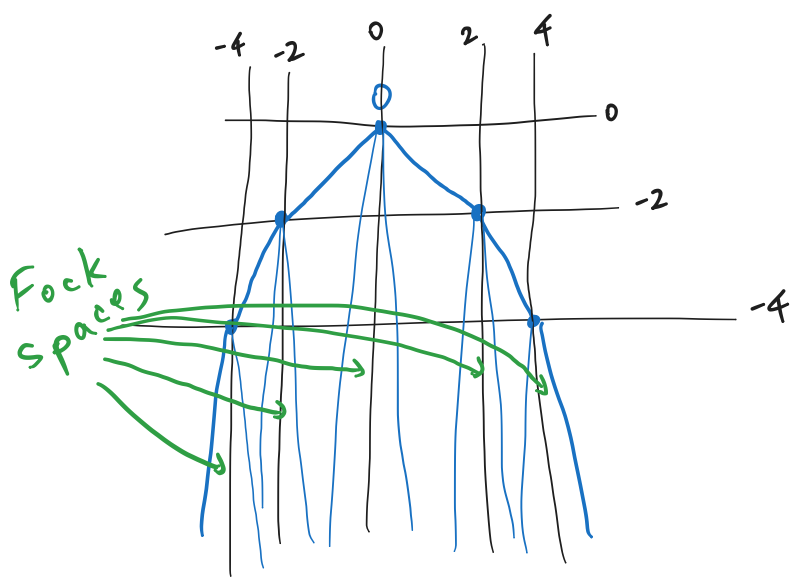
Consider $M(\ell, k)$ (the $\widehat{\mathfrak{sl}_2}$-Verma module with highest weight $\ell$ with level $k$). Let $(\ell, k)$ be generic. Then we can consider

$$M(\ell,k) \otimes L(0,1) = \bigoplus_{m\in\mathbb{Z}} M(\ell-2m, k+1) \otimes \underbrace{\mathbb{V}(h,c)}_{\text{representation of } Vir, \cong \text{ Fock space}}$$

is a direct sum of Vermas with highest weights being the highest weight of $M(\ell, k)$ (namely $\ell$ and some weight of $L(0,1)$).
Recall that

$$\mathrm{ch}_{L(0,1)} = \frac{\Theta_{0,1}}{\varphi(q)} = \frac{1}{\prod_{n=1}^{\infty}(1-q^n)} \sum z^{2m} q^{n^2}$$

This means that we have the picture



**Proposition 218.**

$\mathbb{V}(h, c)$ is the Verma module over $Vir$ with highest weight $h$, central charge $c$, given by

$$c = 1 - \frac{6}{(k+2)(k+3)}, \qquad h = m^2 + \frac{\ell(\ell+2)}{4(k+2)} - \frac{(\ell-2m)(\ell-2m+2)}{4(k+3)}.$$

***Proof.***
The highest degree component of $\mathbb{V}(h, c)$ with respect to $d$ comes from

$$\underbrace{v_{\ell,k}}_{\text{highest, } d=0} \otimes \underbrace{v_{-m,1}}_{\text{extremal, } d=-m^2} + \dots.$$

If we compute the eigenvalue of $L_0$ on this space, we get $\frac{0\cdot(0+2)}{(1+2)\cdot 4} = 0$.
The central charge is

$$c = \frac{3k}{k+2} + \frac{3\cdot 1}{1+2} - \frac{3(k+1)}{k+1+2} = 1 - \frac{6}{(k+2)(k+3)}.$$

□

So if we pick generic parameters, then we get generic modules for Virasoro. If we pick something more special, we can get more special and smaller modules. Next time we will see how these modules will give us Shapovalov determinants.

# Dec 4

## Kac formula for Shapovalov determinant for $Vir$

We consider the Verma module $\mathbb{V}(h, c)$ for the Virasoro algebra, where $c$ is the central charge and $h$ is the highest weight. In other words, $\mathbb{V}(h, c)$ is generated by the highest vector $v_{h,c}$ which is annihilated by all positive operators $L_{>0}v_{h,c} = 0$, and $L_0 v_{h,c} = h \cdot v_{h,c}$, and also the central element acts by $c$, i.e.

$$[L_i, L_{-i}]v_{h,c} = \left(2ih + \frac{i^3 - i}{12} \cdot c\right)v_{h,c}.$$

There are no further relations. Therefore, as vector spaces, $\mathbb{V}(h, c)$ is just $\mathcal{U}(Vir_-)v_{h,c}$, where $Vir_- = \{L_i \mid i < 0\}$. In particular, the character of $\mathbb{V}(h, c)$ is given by

$$\mathrm{Tr}_{\mathbb{V}(h,c)} q^{L_0} = \mathrm{ch}_{\mathbb{V}(h,c)} = \frac{q^h}{\varphi(q)}, \qquad \varphi(q) = \prod_{n=1}^{\infty}(1 - q^n).$$

It turns out that the determinant of the Shapovalov form controls whether this Verma module is irreducible or not. Recall that $\mathcal{D}_n$ is the determinant of the contravariant form on the weight space $\mathbb{V}(h,c)_n = \{v \in \mathbb{V}(h,c) \mid L_0 v = (h+n)v\}$ (for $n \in \mathbb{Z}_{\geq 0}$). Also note that $\dim \mathbb{V}(h,c)_n = \mathcal{P}(n)$, the partition number (i.e. number of partitions of $n$).

**Theorem 219.**

$\mathbb{V}(h,c)$ is irreducible $\iff$ $\mathcal{D}_n \neq 0$ for all $n$.
$\mathcal{D}_n = 0 \implies$ there exists a proper submodule $N \subset \mathbb{V}(h,c)$ such that $N \cap \mathbb{V}(h,c)_n \neq 0$.

***Proof.***
What we already know:

1. $\mathcal{D}_n \in \mathbb{C}[h,c]$, so it's a polynomial in $h$ and $c$.
2. We have an estimate for the degree of this polynomial: $\deg \mathcal{D}_n \leq d_n$, where $\sum_n d_n t^n = \frac{\partial}{\partial u} \prod_{n=1}^{\infty} \frac{1}{1-ut^n}\Big|_{u=1}$.
3. Now imagine the Verma module by the standard "downwards cone" picture. Suppose some $\mathcal{D}_n$ vanishes for some $h, c$. Then there is a submodule with nontrivial intersection with the corresponding weight space $\mathbb{V}(h,c)_n$. Then any nonzero vector in this intersection freely generates a module over $\mathcal{U}(Vir_-)$. This proper submodule contains a space of the same size as the Verma module, but starting at a point in the weight space $n$, rather than at the top. So any factor arising at this top level of the submodule arises also in all of the levels under it, with multiplicity $\mathcal{P}(1)$, then $\mathcal{P}(2)$, then $\mathcal{P}(3)$, and so on.

If we assume that all of the "new" factors do not have common divisors, then the degrees of the new factors are

$$\leq \frac{\frac{\partial}{\partial u} \prod_{n=1}^{\infty} (1-ut^n)^{-1}|_{u=1}}{\prod_{n=1}^{\infty}(1-t^n)} = \frac{\partial}{\partial u}\Big|_{u=1} \log \prod_n (1-t^n) = \sum_{n=1}^{\infty} \frac{t^n}{1-t^n}.$$

So our **naive expectation** is that in $\mathcal{D}_n$, we have new linear factors iff $n = r \cdot s$, $r, s \in \mathbb{Z}_{>0}$. The reality is not quite true (as seen on the homework).

The less naive expectation is that there are quadratic factors, corresponding to unordered pairs $(r,s)$, and the linear factors corresponding to $n = r^2$ (i.e., pairs where $r = s$).

It turns out this less naive expectation is correct! We can prove this using GKO construction.
**Plan:** guess the quadratic factors $Q_{r,s}$ and give $N \subsetneq \mathbb{V}(h,c)$ such that $N \cap \mathbb{V}(h,c)_n \neq 0$. It will turn out that for infinitely many $(h,c)$, that $Q_{r,s}(h,c) = 0$.

We can regard the $N$ as follows. There exists a highest weight Virasoro module $\mathcal{U}$ with highest weight $(h,c)$ such that the dimension of the $n$th weight space $\dim \mathcal{U}_n < P(n)$. We will extract this from the tensor product rule for the $\widehat{\mathfrak{sl}_2}$-modules. Recall that the Virasoro Verma modules appeared as multiplicity spaces in the tensor product. We can compute their highest weights and characters, and it appears that they satisfy this property.

□

## Integrable $\widehat{\mathfrak{sl}_2}$-modules $L(\ell, k)$

Here, $\ell, k \in \mathbb{Z}_{\geq 0}$, $\ell \leq k$, and $\ell$ denotes the highest weight while $k$ denotes the level.
We have

$$L(\ell,k)\otimes L(0,1)=\bigoplus \underbrace{\overbrace{\mathcal{U}(h,c)}^{Vir\curvearrowright}}_{d-\text{graded multiplicity spaces}}\otimes L(\ell-2m,k+1).$$

We have formulas

$$\begin{aligned} c &= 1-\frac{6}{(k+2)(k+3)}, \\ h &= m^2+\frac{\ell(\ell+2)}{4(k+2)}-\frac{(\ell-2m)(\ell-2m+2)}{4(k+3)}. \end{aligned}$$

We also have character formulas

$$\begin{aligned} \operatorname{ch}\mathcal{U}(h,c) &= \frac{f_m^{k,\ell}(q)-f_{\ell+1-m}^{k,\ell}(q)}{\varphi(q)}, \\ f_m^{k,\ell} &= \sum_j q^{(k+2)(k+3)j^2+((\ell+1)+2m(k+2))j+m^2}. \end{aligned}$$

The character formula comes from the Kac-Peterson product rule for Θ.
Now if we make a change of variables $r=\ell+1$, $s=\ell+1-2m$, then we get

$$\begin{aligned} h &= \frac{((k+3)r-(k+2)s)^2-1}{4(k+2)(k+3)}, \\ &= h_{r,s}^{(k)} = \frac{1}{48}((13-c)(r^2+s^2)+\sqrt{(c-1)(c-25)}(r^2-s^2)-24rs-2+2c), \\ \operatorname{ch}\mathcal{U}(h,c) &= \frac{q^{h_{r,s}^{(k)}}}{\varphi(q)}\left(1-q^{rs}-q^{(k+2-r)(k+3-s)}+\sum q^{\text{higher degree terms}}\right). \end{aligned}$$

The character formula implies that

$$\dim \mathcal{U}(h,c)_n < P(n),$$

where $n=r\cdot s$.

**Proposition 220.**

If $h=h_{r,s}(c)$, then $\mathcal{D}_n=0$.

**Theorem 221.**

Let $h_{r,s}(c)$ be the $h^{(k)}_{r,s}$ as defined above. For $r \neq s$ let $Q_{r,s}(h,c) = (h - h_{r,s})(h - h_{s,r})$, a quadratic polynomial in $h, c$. For $r = s$ define $Q_{r,r} = h + \frac{(r^2-1)(c-1)}{24}$. Then

$$\mathcal{D}_n = \underbrace{\text{constant}}_{\neq 0} \cdot \prod_{1 \leq s \leq r \leq n} Q_{r,s}^{P(n-rs)}.$$

***Proof.***
We know that $\mathcal{D}_n$ is divisible by the right hand side, and $Q_{r,s}(h,r)$ have no common factors.

□

## Recap of singular vectors

The determinants of the Shapovalov form control when singular vectors appear in the Verma module. More precisely, for a Verma module $M(\lambda)$, a singular vector appears in a given weight space precisely when the corresponding Shapovalov determinant introduces a new factor which vanishes for that value of $\lambda$.

For the classical finite-dimensional $\mathfrak{g}$ and the affine Kac-Moody $\widehat{\mathfrak{g}}$ (and more generally, for any Kac-Moody Lie algebras, not to be confused with affine Kac-Moody Lie algebras which are a special type of Kac-Moody Lie algebras; the more general Kac-Moody Lie algebras are just Lie algebras arising from certain Cartan matrices, and therefore encompass both affine Kac-Moody Lie algebras and the finite-dimensional classical Lie algebras), the triangular weight decomposition implies that the Shapovalov determinants always decompose into a product of linear factors. As such, the vanishing of *any* of the Shapovalov determinants in $M(\lambda)$ corresponds to $\lambda$ lying on a union of hyperplanes. In the finite-dimensional simple case, these hyperplanes are precisely the affine root hyperplanes. For example, in the $\mathfrak{sl}_3$ case, we have the familiar picture:

“32-Figure1.3-1.png” could not be found.

So the condition that $M(\lambda)$ has a singular vector precisely corresponds to $\lambda$ lying on one of these hyperplanes.

Now, when $\lambda$ lies on the intersection of hyperplanes in the dominant chamber (i.e. it is a dominant integral weight), then in fact it has *two* singular vectors generating two Verma submodules. These are then enough to make the quotient finite-dimensional, giving us the usual picture of irreducible finite-dimensional $\mathfrak{sl}_3$-modules. The affine Kac-Moody case is completely analogous.

**Remark 222.**

While the picture is quite simple, the details can be rather ugly. In particular, the singular vectors corresponding to hyperplanes corresponding to simple roots (in the above picture, this would be the blue and purple lines, and shifts of them) are significantly "simpler" than the singular vectors corresponding to hyperplanes corresponding to positive, but not simple, roots. This is because singular vectors corresponding to hyperplanes of simple roots are generated directly from the highest weight vector by the corresponding monomial in the PBW basis. On the other hand, singular vectors corresponding to hyperplanes of positive, but not simple, roots are instead linear combinations of the monomial action on the highest weight vector inside the correct weight space. It is not a direct "monomial" itself.

For the Virasoro algebra, the picture is slightly more complicated. For $\mathbb{V}(h, c)$, we have Shapovalov determinants which again give conditions for the singular vectors to arise. However, these arise for $h = h_{r,s}$, and depending on $r, s$ these may give quadratic or linear relations on $h, c$. As a result, the condition for $\mathbb{V}(h, c)$ to have submodules is equivalent to $(h, c)$ lying on some union of hyperbolas and lines.

# Dec 6

## Minimal Virasoro modules and classification problem for unitary representations

We'll discuss these two topics, in that order. (We'll at least start on the classification problem for unitary representations.)

### Category-$\mathcal{O}$ Virasoro modules

Last time, we constructed category-$\mathcal{O}$ Virasoro modules which are "extremely small," which were quotients of the Verma module by the submodule generated by two independent vectors:

$$\mathbb{V}(h, c)/\left(\mathbb{V}(h', c) + \mathbb{V}(h'', c)\right).$$

**Example 223.**

The GKO construction:

$$\begin{aligned} c &= 1 - \frac{6}{(k+2)(k+3)} \\ h = h_{r,s} &= \frac{((k+3)r - (k+2)s)^2 - 1}{4(k+2)(k+3)} \end{aligned}$$

for $1 \leq r, s \leq k+1$.

The idea for constructing minimal Virasoro modules is to generalize the above example, by making $\underbrace{k+2}_{p\backslash}, \underbrace{k+3}_{p}$ independent parameters.

We have $\frac{pp'}{(p-p')^2} = (k+2)(k+3)$. We get

$$
\begin{aligned}
h_{r,s} &= \frac{(pr - p's)^2 - (p-p')^2}{4pp'}, \\
s &= 1 - \frac{6(p-p')^2}{pp'},
\end{aligned}
$$

neither of which change under scaling $(p, p') \mapsto (\lambda p, \lambda p')$.

**Generalization:** can assume $p, p'$ are any pair of coprime integers. Now let $1 \leq r < p'$, $1 \leq s < p$.

**Proposition 224.**

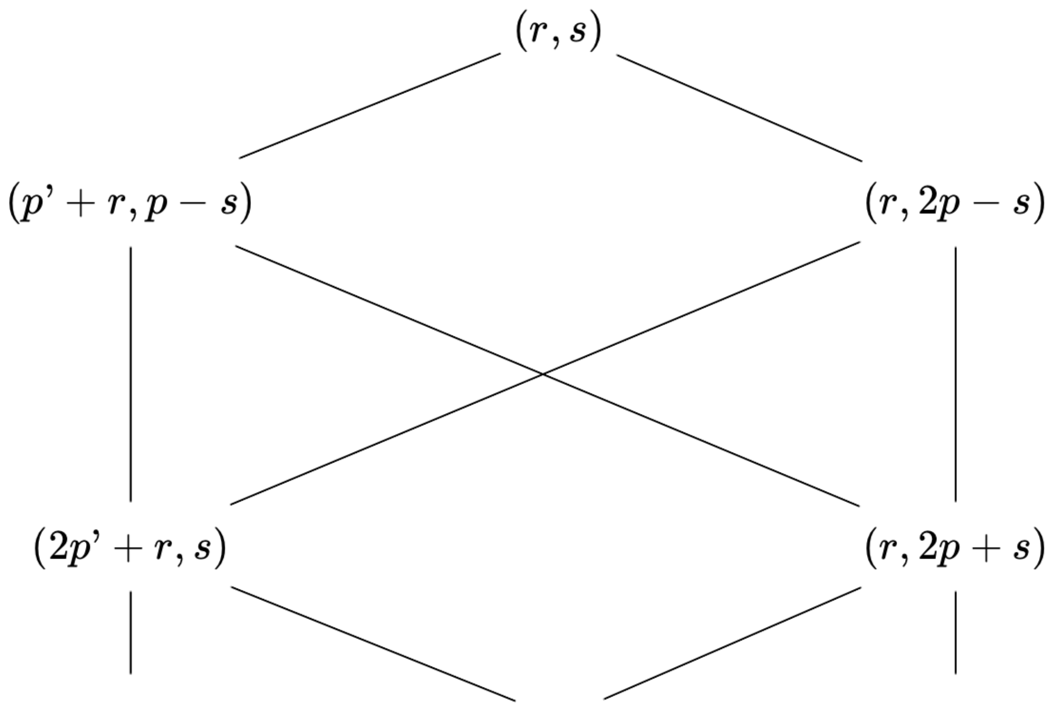
For $c$ and $h = h_{r,s}$ as above written in terms of $p, p'$, $\mathbb{V}(h_{r,s}, c)$ has a singular vector:

1. of the weight $h_{r,s} + rs$,
2. of the weight $h_{r,s} + (p'-r)(p-s)$ (because $h_{r,s} = h_{p'-r,p-s}$).

**Observation:**

$$
\begin{aligned}
h_{r,s} + rs &= h_{p'+r,p-s} = h_{p'-r,p+s}, \\
h_{r,s} + (p'-r)(p-s) &= h_{r,2p-s} = h_{2p'-r,s}.
\end{aligned}
$$

This means that the Verma submodule generated by the singular vector of weight $h_{r,s} + rs$ also has two different singular vectors; same for $h_{r,2p-s} = h_{2p'-r,s}$. This is aptly encoded below: the Verma module associated to $(r, s)$ has two submodules associated to $(p'+r, p-s)$ and $(r, 2p-s)$, and so on: one can iterate this procedure to get infinitely many submodules in the original Verma module.




This is actually quite a nontrivial statement - that the submodules form the lattice shown above! For example, it could happen that there are multiple singular vectors of a specified weight. And if there are multiple singular vectors of a given weight, then for example, the submodules associated to two different weights which both share a submodule (themselves) of the same weight, might not intersect anymore (for example, the submodules of highest weights $(p' + r, p + s)$ and $(r, 2p - s)$ both have a submodule of highest weight $(2p' + r, s)$, but if there were multiple singular vectors of weight $(2p' + r, s)$ then these two submodules might not even intersect). However, it turns out that the picture is as above, and the proof doesn't even use much beyond the computation of the determinant of the Shapovalov form.

**Theorem 225** (Feigin-Fuks)**.**

If $c, h$ are as above, then all submodules in $\mathbb{V}(h, c)$ are generated by singular vectors, and all Verma submodules are described by the picture:

$$\begin{tikzcd}
& (r,s) \arrow[dl, dash] \arrow[dr, dash] & \\
(p'+r, p-s) \arrow[d, dash] \arrow[drr, dash] & & (r, 2p-s) \arrow[d, dash] \arrow[dll, dash] \\
(2p'+r, s) \arrow[d, dash] \arrow[dr, dash] & & (r, 2p+s) \arrow[d, dash] \arrow[dl, dash] \\
{} & {} & {}
\end{tikzcd}$$

We’re not going to prove this, however. Let’s switch subjects now.

## Unitary representations

All of the infinite-dimensional Lie algebras we have considered so far, namely $\mathfrak{a}, \widehat{\mathfrak{g}}, Vir$, are defined over $\mathbb{R}$. Furthermore, the antiautomorphism $\Theta : \mathcal{U}(L) \to \mathcal{U}(L)^{\mathrm{op}}$ is defined over $\mathbb{R}$ as well. This means we can extend it to an antilinear automorphism defined over $\mathbb{C}$. This means we can define a Hermitian version of the Shapovalov form.

**Proposition 226.**

Such a $\langle , \rangle$ exists on any Verma module with real highest weight.

**Definition 227.**

An $L$-module $M$ is called **unitary** if there exists a positive-definite contravariant Hermitian $\langle , \rangle$.

**Remark 228.**

On the Lie algebra itself, the $\langle , \rangle$ is almost never positive-definite.

**Proposition 229.**

A unitary $M$ is semisimple.

***Proof.***
Any submodule has an orthogonal complement.

□

**Example 230.**

Consider $L(\lambda)$ (simple quotient module) for $\mathfrak{sl}_2$ and $\lambda \in \mathbb{R}$.

**Proposition 231.**

$L(\lambda)$ is unitary $\iff \lambda \in \mathbb{Z}_{\geq 0}$.

***Proof.***
Let $v_\lambda$ be the highest weight vector. Then $L(\lambda)$ is spanned by $v_\lambda, fv_\lambda, f^2v_\lambda, \ldots$ We can compute that $ef^kv_\lambda = c_k \cdot f^{k-1}v_\lambda$ for $c_k = (\lambda - 2k) + c_{k-1} = k\lambda - k(k+1)$ which is negative for $k \gg 0$ . Then we compute that $\langle f^kv_\lambda, f^kv_\lambda \rangle = \langle v_\lambda, e^kf^kv_\lambda \rangle < 0$ so long as none of the $c_k = 0$ and $k \gg 0$, as the form ends up changing sign, eventually giving a negative.

□

**Corollary 232.**

$L(\lambda)$ for $\widehat{\mathfrak{sl}_2}$ can be unitary only if $\lambda$ is dominant integral.

**Theorem 233.**

In fact $L(\lambda)$ for $\widehat{\mathfrak{sl}_n}$ is unitary iff $\lambda$ is dominant integral.

We'll do this next time.

**Example 234.**

The Fock space $F_\mu$ for $\mathfrak{a}$ (recall that $\mu$ is the eigenvalue of $a[0] \in \mathfrak{a}$).

**Proposition 235.**

For $\mu \in \mathbb{R}$, $F_\mu$ is unitary.

***Proof.***
The monomials $\prod a[-r_i]^{k_i} \cdot v_\mu$ are orthogonal. So for such a vector, the inner product with itself is $\prod r_i^{k_i} > 0$.

□

**Corollary 236.**

We get unitary Virasoro modules:

$$L_n := \frac{1}{2} \sum_m \quad : a_n a_{n+m} : \quad + i\lambda n a_n \quad \curvearrowright F_\mu$$

for $n \neq 0$, $L_0 := \frac{\mu^2+\lambda^2}{2} + \sum_{m>0} a_{-m} a_m$, and $\Theta(L_n) = L_{-n}$.
The highest weight is $c = 1 + 12\lambda^2$.

**Corollary 237.**

The simple $Vir$-modules $L(h, c)$ is unitary if $c \geq 1$ and $\lambda \geq c - 1$.

# Dec 11

## Unitarity for representations of $\widehat{\mathfrak{sl}_n}$ and $Vir$

First, let's consider $L(\widehat{\lambda})$, an irreducible highest weight $\widehat{\mathfrak{g}}$-module.
**Necessary condition:** $L(\widehat{\lambda})$ is unitary $\implies$ $\widehat{\lambda}$ is a dominant integral weight $\iff$ $\langle \widehat{\lambda}, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}$ for all simple roots $\alpha_i$.

**Proposition 238.**

The converse is true: $\widehat{\lambda}$ is dominant integral weight $\implies$ $L(\widehat{\lambda})$ is unitary.

***Proof.***

We will only prove this for $\widehat{\mathfrak{sl}_n}$, because there is a really nice construction, even though it holds for all Kac-Moody algebras.

Let $\mathfrak{g} = \mathfrak{sl}_n$. First, it's sufficient to show for fundamental representations, which for $\widehat{\mathfrak{sl}_n}$ are just $L(0,1)$ and $L(\omega_i, 1)$. For an arbitrary dominant weight $\widehat{\lambda} = \sum_{i=0}^{n-1} n_i \widehat{\omega}_i$ for $n_i \in \mathbb{Z}_{\geq 0}$ (where $\widehat{\omega}_0 = (0,1)$ and $\widehat{\omega_i} = (\omega_i, 1)$ for $1 \leq i \leq n-1$), we have an embedding $L(\widehat{\lambda}) \hookrightarrow \bigotimes_i L(\widehat{\omega}_i)^{\otimes n_i}$. In particular, if we know that each of the $L(\widehat{\omega}_i)$ are unitary, then the tensor product is as well, and hence the restriction to $L(\widehat{\lambda})$ is also unitary.

Next, we use the "fermion" construction of $L(*, 1)$. Namely, consider the infinite-dimensional vector space $\mathbb{C}^n[z, z^{-1}]$, which carries an action by $\mathfrak{sl}_n[z, z^{-1}] \subset \mathfrak{gl}_\infty^J$, the generalized Jacobi matrices, which in turn also act on $\mathbb{C}^n[z, z^{-1}]$. For this action, we need some ordering of a basis on $\mathbb{C}^n[z, z^{-1}]$, by taking some basis $v_1, \ldots, v_n$ of $\mathbb{C}^n$ and then taking the ordered basis $\ldots, z^{-1}v_n, v_1, \ldots, v_n, zv_1, \ldots, zv_n, z^2v_1, \ldots$. Now this representation of the loop algebra will give a representation of its central extension $\widehat{\mathfrak{sl}_n}$ on the semi-infinite wedge space.

As a result, we get an action $\widehat{\mathfrak{sl}_n} \curvearrowright \Lambda^{\frac{\infty}{2}+\bullet}(\mathbb{C}^n[z, z^{-1}])$. We can regard this space as an irreducible representation of the Clifford algebra $C\ell(\psi_i^j, \psi_i^{j*})$ for $j = 1, 2, \ldots, n$ and $i \in \mathbb{Z}$. It is generated by the vector

$$\bigwedge_{j=1}^{n} \bigwedge_{i=0}^{\infty} \psi_i^j,$$

where $\psi_i^j$ acts by $\psi_i^j \wedge -$ and $\psi_i^{j*}$ acts by $\partial_{\psi_{-i}^j}$. This construction is analogous to the construction at the beginning of the course, and the action of $\widehat{\mathfrak{sl}_n}$ here is analogous to the action of the Heisenberg algebra.

We can compute the level. $h_1[m]$ acts as $: \sum_{r+s=m} \psi_r^1 \psi_s^{1*} - \psi_r^2 \psi_s^{2*} :$. Then we can compute that

$$\begin{aligned} [h_1[m], h_1[-m]] &= \underbrace{2m} \cdot \mathrm{id}, \\ &= \langle h_1, h_1 \rangle \cdot m \cdot k, \\ &\implies k = 1. \end{aligned}$$

**Proposition 239.**

1. $\Lambda^{\frac{\infty}{2}+\bullet}(\mathbb{C}^n[z, z^{-1}])$ is integrable.
2. There exists a nontrivial homomorphism $L(\widehat{\omega_i}) \to \Lambda^{\frac{\infty}{2}+\bullet}(\mathbb{C}^n[z, z^{-1}])$.
3. $\Lambda^{\frac{\infty}{2}+\bullet}(\mathbb{C}^n[z, z^{-1}])$ is unitary.

***Proof.***

1. We have to show that $e_i, f_i$ for $i = 0, \dots, n-1$ act locally nilpotently. In fact, we can show this for any $e_{ij}[m]$ for $i \neq j$, for any Fourier component of any non-Cartan part. The element $e_{ij}[m]$ acts as $:\sum_{r+s=m} \psi_r^i \psi_s^{j*}:$. We can represent any monomial by $n$ horizontal number lines, and on each number line we have some markings on the integers which are bounded below and hit every sufficiently large integer. Now the $\psi_r^i \psi_s^{j*}$ takes $s$ in the $j$th line, shifts it by $m = r + s$, then puts it in the $i$th line (in that spot), if possible (i.e. if it's already empty). Since you can only do this finitely many times, the $e_{ij}[m]$ act locally nilpotently. So as a representation of $\widehat{\mathfrak{sl}_n}$, it is integrable.
2. The highest vectors are again given by monomials. In the picture from before, we fill everything to the right of the $0$ line, and put exactly $i$ markings on the zero line starting from the top. This will be a highest vector; it will be annihilated by all positive Chevalley generators, and the highest weight will be $\widehat{\omega}_i$. We get a map from the Verma module $M(\widehat{\omega}_i)$, which must fact through $L(\widehat{\omega}_i)$ since part 1 says that the target is integrable.
3. $\Lambda^{\frac{\infty}{2}+\bullet}(\mathbb{C}^n[z, z^{-1}])$ is a unitary $C\ell$-module, with the monomial basis being the orthonormal one: we have that the dual of $\psi_i^j$ is $\psi_{-i}^{j*}$, which implies that $e_{ij}[m]^* = e_{ji}[-m]$, which is what we needed.

□

This proves everything: we have embeddings $L(\widehat{\omega}_i) \hookrightarrow \Lambda^{\frac{\infty}{2}+\bullet}(\mathbb{C}^n[z, z^{-1}])$, which are unitary, hence the $L(\widehat{\omega}_i)$ are unitary, hence any $L(\widehat{\lambda})$ is unitary for dominant integral $\widehat{\lambda}$.

□

**Corollary 240.**

$$L(\widehat{\lambda}) \text{ is unitary} \iff \widehat{\lambda} \text{ is a dominant weight.}$$

## Consequences for $Vir$

We had a computation on small Virasoro module:

$$L(h_{rs}, c), \quad c = 1 - \frac{6}{(k+2)(k+3)}, \quad k \in \mathbb{Z}_{\geq 0},$$

where $L(h_{rs}, c)$ arises in the following way. We take the tensor product $L(0,1) \otimes L(\lambda, k)$ of $\widehat{\mathfrak{sl}_2}$-representations, and this decomposes with respect to the diagonal $\widehat{\mathfrak{sl}_2}$-action as

$$L(0,1) \otimes L(\lambda, k) = \bigoplus \underbrace{L(h_{r,s}, c)}_{Vir-\text{module, discrete series, Unitary}} \otimes L(r, k+1),$$

where the $L(h_{r,s}, c)$ are called the discrete series Virasoro modules. Once we know that $L(0,1)$ and $L(\lambda, k)$ are unitary representations of $\widehat{\mathfrak{sl}_2}$, then we know that the multiplicity spaces $L(h_{r,s}, c)$ are unitary $Vir$-modules. The reason for this is as follows. We have

$$L_n = \underbrace{L_n}_{\text{Sugawara}} \otimes 1 + 1 \otimes \underbrace{L_n}_{\text{Sugawara}} - \Delta(\underbrace{L_n}_{\text{Sugawara}}),$$

where

$$\text{Sugawara}\, L_n = \quad : \sum_{r+s=n} x_a[r] x^a[s] :.$$

We then conclude that the adjoint of $L_n$ is $L_{-n}$ by computing the adjoints of each term in the sum and reversing the order; this implies that all of the $L(h_{r,s}, c)$ are unitary Virasoro modules.

Now let's summarize what we already know about unitarity for Virasoro modules. The answer here is much more complicated than in the case for Kac-Moody algebras, where unitarity is basically the same as integrability.

$L(h, c)$ for $h \geq 0$, $c \geq 1$ is unitary. This comes from the Kac formula for determinant of the Shapovalov form. What we want is to determine if this is always $\geq 0$ or not.

$$\begin{aligned}
\mathcal{D}_n =& C \cdot \prod_{rs \leq n} \varphi_{r,s}^{P(n-rs)}(h, c), \\
\varphi_{r,s}(h, c) =& \left(h - \frac{(r-s)^2}{4}\right)^2 + \frac{h}{24}(r^2+s^2-2)(c-1) + \frac{1}{576}(r^2-1)(s^2-1)(c-1)^2 \\
&+ \frac{1}{48}(c-1)(r-s)^2(rs+1), \\
\varphi_{r,r}(h, c) =& h + \frac{(r^2-1)(c-1)}{24}.
\end{aligned}$$

Although these formulas are rather long, at least they can be computed. This is already maximally simplified.

One can observe that in the region $h \geq 0$ and $c \geq 1$, then $\varphi_{r,s}(h, c) \geq 0$, and it is almost all linear. We're out of time, but next time (last class) we'll explain: if $c < 0$ or $h < 0$, then $L(h, c)$ can't be unitary (for $\mathfrak{sl}_2$ the reason for this is just a computation); and also we will see the most interesting case, which is for $h \geq 0$ and $0 \leq c < 1$. It will turn out that in the region $0 \leq c < 1$, the only unitary representations are $L(h_{r,s}, c)$ as described under Consequences for $Vir$, and in fact we see that the central charge is always between $0$ and $1$. It turns out that this is a complete answer on the unitarity problem for Virasoro. But this is very difficult, so we won't see a complete proof, but at least we will see some of the ideas.

# Dec 13

This is the last class.

## Unitarity of $Vir$ modules

By now, we have some known cases where the irreducible Virasoro module is unitary.

1. from boson realization of $Vir$. For $n \neq 0$ write

$$L_n := \frac{1}{2} : \sum_{m \in \mathbb{Z}} a_{-m} a_{n+m} : \quad + i\lambda n a_n \qquad \curvearrowright \quad F_\mu,$$

acting on the Fock space, where $a_n$ are generators of the Heisenberg algebra $\mathfrak{a}$ are such that the highest vector $v_\mu \in F_\mu$ is annihilated by all $a_{>0}$ and $a_0 v_\mu = \mu \cdot v_\mu$, and also $[a_n, a_{-n}] = n$.
Then $F_\mu$ is a unitary representation of the Heisenberg algebra $\mathfrak{a}$ once $\mu \in \mathbb{R}$. Moreover, the $L_n$ according to the formula above generate the Virasoro action $Vir \curvearrowright F_\mu$, and since we have the factor of $i$, we even have the adjoint of $L_n$ is $L_{-n}$, i.e. $L_n^\intercal = L_{-n}$. So $F_\mu$ is a unitary representation of $Vir$. We can compute that the central charge is $c = 1 + 12\lambda^2$ and the highest weight is $h = \frac{\mu^2}{2} + \frac{\lambda^2}{2}$. This implies that $L(h, c)$ is a unitary $Vir$-module. This construction gives $(h, c)$ where $c > 1$ and $h \geq \frac{c-1}{24}$.
2. From GKO construction. We have $Vir$-modules $L(h_{r,s}, c)$; if $c = 1 - \frac{6}{(k+2)(k+3)}$ for $k \in \mathbb{Z}_{\geq 0}$, so that $0 < c < 1$, then this is unitary. Recall that $c$ is the central charge (eigenvalue of central element) and $h$ is the eigenvalue of $L_0$ on the highest weight vector.

**Theorem 241.**

1. $L(h, c)$ is always unitary for $c \geq 1$, $h \geq 0$.
2. $L(h, c)$ cannot be unitary for $c < 0$ or $h < 0$.
   This leaves only the case $0 \leq c < 1$ and $h \geq 0$, which is quite difficult.

***Proof.***

1. The Kac formula for the Shapovalov determinant is as follows.

$$\begin{aligned}
\mathcal{D}_n =& \underbrace{C}_{\text{positive integer}} \prod_{\substack{r \geq s, \\ rs \leq n}} \varphi_{r,s}^{P(n-rs)}, \\
\varphi_{r,r} =& h + \frac{(r^2-1)(c-1)}{24}, \\
\varphi_{r,s} =& \left(h - \frac{(r-s)^2}{4}\right)^2 + \frac{h}{24}(r^2+s^2-2)(c-1) \\
&+ \frac{(r^2-1)(s^2-1)(c-1)^2}{576} + \frac{(c-1)(r-s)^2(rs+1)}{48}.
\end{aligned}$$

Recall that $\mathcal{D}_n$ is only defined up to a constant, but upon further reflection the constant scalar is a positive integer. We have $\mathcal{D}_n > 0$ for $c \geq 1$ and $h \geq 0$, unless $c - 1 = 0 = h$.

Let's first examine the situation when $h > 0$. Then in the region $c \geq 1$ and $h > 0$, the Hermitian form $\langle , \rangle$ on the $n$th weight space of the Verma module $\mathbb{V}(h, c)$ has constant signature, because the determinant is always positive (and the signature can only change when the Hermitian form becomes non-degenerate, i.e. when the determinant becomes $0$). This implies that to show $L(h,c)$ is unitary, it's sufficient to give at least one pair $(h,c)$ in the region $h \geq 0$ and $c \geq 1$ where $\langle , \rangle$ is positive definite on $\mathbb{V}(h,c)_n$. This can be seen to be true if $h > \frac{c-1}{24}$, the bosonic example give above. This means that for $h > 0$, the form is non-degenerate on the Verma module, so $\mathbb{V}(h,c) = L(h,c)$ irreducible and it is unitary. We still have to address the case $h = 0$. But in this case, the only existing contravariant Hermitian form is positive semidefinite. Moreover, $L(h,c) = \mathbb{V}(h,c)/\mathrm{Rad}\langle , \rangle$, so the form is positive definite on the quotient, and hence $L(h,c)$ is indeed unitary (although now it is a quotient of $\mathbb{V}(h,c)$, and no longer equal to the entire Verma module).

2. Take a highest weight vector $v \in L(h,c)$ such that $L_{>0}v = 0$, $L_0 v = hv$, and $\langle v, v\rangle = 1$. Then we compute:

$$\begin{aligned}\langle L_{-n}v, L_{-n}v\rangle &= \langle v, L_n L_{-n} v\rangle, \\ &= \langle v, [L_n, L_{-n}]v\rangle, \\ &= \left(2nh + \frac{n^3 - n}{12}c\right).\end{aligned}$$

If $c < 0$ then picking $n \gg 0$ we get a negative value, hence $L(h,c)$ is not unitary. If $h < 0$ then picking $n = 1$ we get a negative value, so $L(h,c)$ is not unitary.

□

**Theorem 242.**
For $0 \leq c < 1$ and $h \geq 0$, the only unitary $L(h,c)$ are $L(h_{r,s}, c)$ where $c = 1 - \frac{6}{(k+2)(k+3)}$ and $h_{r,s}$ is of the particular form in terms of $r$, $s$.

This is very hard, so we won't actually prove it. Instead we'll discuss some ideas that go into the proof.

**Remark 243.**

Recall that the formula for $c$ comes from the GKO construction. In the $\widehat{\mathfrak{sl}_2}$-modules $L(0,1)$ and $L(\lambda, k)$, we can take the tensor and get $L(0,1) \otimes L(\lambda, k) = L(h_{r,s}, c) \otimes L(\mu, k+1)$. If we try to use larger Lie algebras or higher weights, the multiplicity spaces (here, they're the $Vir$-modules $L(h_{r,s}, c)$) are much larger, and may not even finitely-generated.

**Definition 244** (ghost number)**.**

Define the "ghost number" $g_n(h,c)$ to be the number of $-1$s in the Jordan normal form of $\langle , \rangle\big|_{L(h,c)_n}$.

**Lemma 245.**

$g_n(h, c) \leq g_{n+1}(h, c)$.

***Proof.***

Consider $\mathfrak{sl}_2 = \mathrm{Span}\{\underbrace{L_1}_{E}, \underbrace{-2L_0}_{H}, \underbrace{-L_{-1}}_{F}\}$. The anti-involution exchanges $E \leftrightarrow -F$ and $H \leftrightarrow H$.

Decompose $L(h, c) = \bigoplus M(-2(h+n))$ with respect to the $\mathfrak{sl}_2$-action. The weights satisfy $-2(h+n) < 0$. On such $M(\lambda)$ with $\lambda < 0$, then $\langle , \rangle$ is definite. This follows by computing the scalar product of $\langle F^k v_\lambda, F^k v_\lambda \rangle = k!(-\lambda)(\lambda+1)(-\lambda+2)\cdots(\lambda+k-1)\langle v_\lambda, v_\lambda \rangle$.

□

From this, we can deduce a weaker statement:

**Proposition 246.**

Let $U_j = \{(h, c) \mid 0 < c < 1,\ h \geq 0,\ \varphi_{j,1}(h, c) < 0\}$. Then for any $(h, c) \in \bigcup_{j \geq 2} U_j$, we have $L(h, c)$ is *not* unitary.

***Proof.***

The idea is to use Lemma 231 and proceed by induction: $U^{(n)} = \bigcup_{2 \leq j \leq n} U_j \ni (h, c)$, then $\langle , \rangle \big|_{L(h,c)_n}$ is indefinite. This comes from analyzing the factors in the Shapovalov determinant. Namely, we need to show that $\varphi_{r,s}(c, h) > 0$ on $U^{(n)} \setminus U^{(n+1)}$ for $r, s > 1$. For this, we need only examine the intersection of $U_n$ with the boundary of $U_{n-1}$, so we can reduce to the set $U_n \cap \{\varphi_{n-1,1} = 0\}$. On $U_n \cap U_{n-1} \setminus U^{(n-2)}$, we know that $\mathcal{D}_n > 0$. On the other hand, the form is indefinite. This means that we have a decomposition into a direct sum of a positive definite part and a negative (semi)definite part, and in particular, the negative part has dimension at least $2$. On the boundary, the rank of the Hermitian form drops by one, because the multiplicity of the zero is one, according to the Kac formula for the Shapovalov determinant. This means that on the boundary we have the radical of the form, and once we quotient $\mathbb{V}(h, c)$ by the radical $\mathrm{Rad}\langle , \rangle$, then we still have an indefinite form, and hence the quotient $L(h, c)$ is not unitary.

□

The full proof is more difficult, but this gives some of the ideas going into the proof.

That's the end of the course! ■