# Chevalley groups, Bruhat-Tits apartments, and Moy-Prasad filtrations

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18.784 Fall 2021

## 1 "Lie groups" over $p$-adic fields

The study of groups arose from studying symmetries of objects. These symmetries were originally defined to be either a subset of $S_n$, the symmetric group, or a subset of $GL(V)$, the invertible linear operators on some vector space. In general, these corresponded with the mathematicians' interest in studying either finite groups or infinite groups. In our case, we are particularly interested in certain infinite groups which turned out to be historically important in a number of ways: these are the Lie groups, which often encompass the infinitely many symmetries of various objects such as spheres, circles, 3-D space, etc. Some examples of these are $GL_n(\mathbb{R})$, $GL_n(\mathbb{C})$, $SL_n(\mathbb{R})$, $SL_n(\mathbb{C})$, $SO(3)$, and $SU(2)$. However, in this paper we will focus on analogs of some of these groups which are defined over more exotic fields, namely nonarchimedean local fields.

First, we will give an overview of Lie groups and describe the construction of Chevalley groups to obtain their analogues over finite extensions of $\mathbb{Q}_p$. We then define the Bruhat-Tits apartment of a Chevalley group, which is a structure that interacts with the Chevalley group and its Lie algebra. We will ultimately utilize the topology of nonarchimedean local fields and the apartment to construct the Moy-Prasad filtrations, which have direct applications to studying representations of $G$.

### 1.1 Lie groups

The classical Lie groups such as $GL_n$, $SL_n$, $SO_n$, and so on are well-studied. These groups often arise as groups of specific types of matrices: $GL_n$ is the group of invertible matrices, $SL_n$ is the group of matrices with determinant 1, and $SO_n$ is the group of orthogonal matrices with determinant 1. In this section, we review some crucial notions of Lie groups, but it is only meant to be an overview: for a proper treatment, see [KJ08].

**Definition 1.** [KJ08, Def. 2.1] A **real Lie group** is a manifold $G$ such that $G$ is endowed with a group structure, such that multiplication and inversion are smooth maps.

Similarly, we have an analogous definition of complex Lie groups.

**Definition 2.** [KJ08, Def. 2.3] A **complex Lie group** is a complex analytic manifold $G$ such that $G$ is endowed with a group structure, such that multiplication and inversion are analytic maps.

In essence, real and complex Lie groups are groups which are able to be identified with (real or complex analytic) manifolds such as $S^1$. Many of the most well-known examples are the classical Lie groups, such as $GL_n(\mathbb{R}), GL_n(\mathbb{C}), SL_n(\mathbb{R}), SL_n(\mathbb{C}), SO_n(\mathbb{R}), SO_n(\mathbb{C}), Sp_{2n}(\mathbb{R}), Sp_{2n}(\mathbb{C})$, and so on. We will be most interested in complex semisimple Lie groups: those which have no nontrivial connected abelian normal subgroups, such as $SL_n(\mathbb{C})$, $SO_n(\mathbb{C})$, and $Sp_{2n}(\mathbb{C})$.

**Definition 3.** To a Lie group $G$, its Lie algebra $\mathfrak{g}$ is defined to be the tangent space to $G$ at the identity, endowed with a Lie bracket $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which is bilinear, antisymmetric, and satisfies the Jacobi identity

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]].$$

When $G$ is a real Lie group, the Lie algebra will be a real vector space; when $G$ is a complex Lie group, the Lie algebra will be a complex vector space. To see how to compute the Lie algebra of various Lie groups in more detail, see [KJ08]. In fact, the Lie bracket is the tangent space map $T_eG \to T_eG$ induced by conjugation; see [KJ08] for more details. However, in this paper, we will not be too concerned about how the Lie bracket arises, because for all matrix subgroups (such as the classical Lie groups mentioned above), the Lie bracket is given by

$$[x, y] = xy - yx.$$

**Definition 4.** The **Killing form** $K(-,-)$ on $\mathfrak{g}$ is defined to be the bilinear form on $\mathfrak{g}$ given by $K(x, y) = \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y)$, where $\operatorname{ad} x : \mathfrak{g} \to \mathfrak{g}$ is the operator $(\operatorname{ad} x)(y) = [x, y]$.

**Definition 5.** An element $x \in \mathfrak{g}$ is **semisimple** if $\operatorname{ad} x : \mathfrak{g} \to \mathfrak{g}$ is semisimple (i.e., diagonalizable).

**Example 6.** Let $GL_n(F)$ be the general linear group on $F = \mathbb{R}$ or $\mathbb{C}$. Then the Lie algebra $GL_n(F)$ is

$$\mathfrak{gl}_n(F) = \mathrm{Mat}_{n\times n}(F),$$

the $n \times n$ matrices. The Lie bracket is given by $[X, Y] = XY - YX$ for $X, Y$ two $n \times n$ matrices with entries in $F$. The semisimple elements are precisely the diagonal matrices in $\mathfrak{gl}_n(F)$.

**Example 7.** Let $SL_n(F)$ be the special linear group on $F = \mathbb{R}$ or $\mathbb{C}$. The Lie algebra of $SL_n(F)$ is

$$\mathfrak{sl}_n(F) = \{X \mid X \in \mathrm{Mat}_{n\times n}(F), \quad \operatorname{tr} X = 0\},$$

the traceless $n \times n$ matrices. The Lie bracket is again given by $[X, Y] = XY - YX$. The semisimple elements are precisely the diagonal matrices in $\mathfrak{sl}_n(F)$.

For ease of discussion, when we say Lie group, we mean one of the classical Lie groups, which are in particular matrix subgroups. From the Lie group, one can obtain its Lie algebra. The reverse is partially true. The Lie group and its Lie algebra are deeply intertwined via the following fact:

**Proposition 8** ([KJ08], Cor. 3.43)**.** *If a Lie group $G$ is simply connected with Lie algebra $\mathfrak{g}$, then $G = \langle\{\exp(X) \mid X \in \mathfrak{g}\}\rangle$. Any other connected Lie group $G'$ with Lie algebra $\mathfrak{g}$ is of the form $G/Z$ for some discrete normal subgroup $Z \subset G$. (Note that the exponential map converges and is therefore well-defined.)*

In fact, the exp map gives a local homeomorphism between neighborhoods of $0 \in \mathfrak{g}$ and $1 \in G$; additionally, if $G$ is connected, any open neighborhood of $1 \in G$ generates all of $G$. In particular, taking the open neighborhood which is homeomorphic to an open neighborhood of $0 \in \mathfrak{g}$ shows that every element of $G$ can be written as a product of $\exp(X)$ for $X \in \mathfrak{g}$ (even better, for $X$ in a specific open neighborhood of $0 \in \mathfrak{g}$). Furthermore, for a Lie group $G$ with Lie algebra $\mathfrak{g}$, we have a bijection between Lie subgroups of $G$ and Lie subalgebras of $\mathfrak{g}$. For $H \subset G$ a Lie subgroup, the corresponding Lie subalgebra of $\mathfrak{g}$ is simply $\mathfrak{h} = T_1(H) \subset \mathfrak{g}$, the Lie algebra of $H$. How exactly does one recover the Lie subgroup from the Lie subalgebra? The answer: For a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, we can obtain the corresponding Lie subgroup by taking the subgroup generated by the exponentials of all $X \in \mathfrak{h}$, intersected with $G$.

**Notation 1.** *We will call the above process the* ***exp*** $\leftrightarrow$ ***d/dt dictionary****, to move between Lie subgroups of* $G$ *and Lie subalgebras of* $\mathfrak{g}$.

*Remark* 9. The $d/dt$ part comes from the fact that in classical Lie groups, the way to obtain elements in the Lie algebra is from differentiating curves in the Lie group. This follows from a more general process in manifolds: see [Lee13] for more details.

We will primarily be interested in complex semisimple simply connected Lie groups. The complex part just means that we are interested in complex Lie groups, as in Definition 2. We define the notions now.

**Definition 10.** A Lie algebra $\mathfrak{g}$ is **simple** if it is nonzero and has no nonzero subspaces $I$ such that $[I, \mathfrak{g}] = \{[i, x] \mid i \in I, x \in \mathfrak{g}\} = I$ (these are called **ideals**). A Lie algebra $\mathfrak{g}$ is **semisimple** if it is a direct sum of simple Lie algebras. A Lie group is semisimple if its Lie algebra is semisimple.

**Proposition 11** (Cartan's criterion)**.** *A Lie algebra* $\mathfrak{g}$ *is semisimple iff the Killing form is nondegenerate.*

**Example 12.** The Lie groups $SL_n(\mathbb{C})$, $SO_n(\mathbb{C})$, and $Sp_{2n}(\mathbb{C})$ are all simple, and therefore semisimple. The Lie group $GL_n(\mathbb{C})$ is not semisimple.

When $G$ is a complex semisimple Lie group, then $\mathfrak{g}$ has several important properties. By Cartan's criterion, the Killing form $(,)$ is a nondegenerate bilinear form. As a result, it induces an isomorphism $\mathfrak{g} \simeq \mathfrak{g}^*$, its dual space.

**Definition 13.** A **toral** subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a commutative (i.e., the Lie bracket is 0 when restricted to $\mathfrak{h}$) subalgebra consisting of only semisimple elements. A **Cartan** subalgebra is a toral subalgebra $\mathfrak{h}$ such that $\mathfrak{h} = \{x \mid [x, \mathfrak{h}] = 0\}$.

Cartan subalgebras turn out to arise as maximal toral subalgebras, and therefore always exist. They are also commonly called a **maximal torus** of $\mathfrak{g}$, and their corresponding Lie subgroup are commonly called a **maximal torus** of $G$. For this reason, Cartan subalgebras are sometimes denoted $\mathfrak{t}$, with the corresponding maximal torus in $G$ denoted by $T$. The reasoning for this name is that in complex semisimple Lie groups (or more generally, split reductive groups), the maximal torus $T \subset G$ splits as a direct product of copies of $\mathbb{C}^\times$. In particular, this means that $T \cong \mathbb{G}_m^n$ for some integer $n$, which implies that $T$ is isomorphic to a torus, which is just defined to be a direct product of copies of $\mathbb{G}_m$. The condition for the Cartan subalgebra implies that $T$ is maximal, hence the name maximal torus.

**Example 14.** In $\mathfrak{gl}_n(\mathbb{C})$ and $\mathfrak{sl}_n(\mathbb{C})$, a Cartan subalgebra is given by the diagonal matrices inside each Lie algebra. Their corresponding maximal tori in $GL_n(\mathbb{C})$ and $SL_n(\mathbb{C})$ are, once again, the diagonal matrices inside each Lie group. Notice that the maximal torus in $GL_n(\mathbb{C})$ turns out to be $(\mathbb{C}^\times)^n$, as each diagonal entry acts independently of each other. On the other hand, the maximal torus in $SL_n(\mathbb{C})$ turns out to be $(\mathbb{C}^\times)^{n-1}$, as the last entry is determined by the previous $n-1$, which are freely chosen and act independently of each other.

Crucially, we have the following important decomposition.

**Theorem 15.** *For* $\mathfrak{g}$ *a complex semisimple Lie algebra with a fixed Cartan subalgebra* $\mathfrak{h}$*, we have that*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^* \setminus \{0\}} \mathfrak{g}_\alpha,$$

*where* $\mathfrak{g}_\alpha = \{x \mid [h, x] = \alpha(h)x\}$*. Setting* $\Phi := \{\alpha \mid \mathfrak{g}_\alpha \neq 0\}$*, we call* $\Phi$ *the* ***root system*** *of* $\mathfrak{g}$*. We have that* $\Phi$ *is finite and each* $\mathfrak{g}_\alpha$ *is one-dimensional. Furthermore, for each* $\alpha, \beta \in \Phi$*, we have* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.

Letting $(,)$ be the Killing form on a complex semisimple Lie algebra $\mathfrak{g}$, then the restriction of $(,)$ to a Cartan subalgebra $\mathfrak{h}$ is again semisimple. As a result, it again induces an isomorphism $\mathfrak{h} \simeq \mathfrak{h}^*$. Using this isomorphism, we obtain a nondegenerate bilinear form $(,)$ on $\mathfrak{h}^*$ as well, and in particular, we can apply it to $\Phi \subset \mathfrak{h}^*$. We will freely use this same notation $(,)$ to mean both the nondegenerate bilinear form on $\mathfrak{h}$ and $\mathfrak{h}^*$.

**Proposition 16.** *For any $\alpha, \beta \in \Phi$, we have that $(\alpha, \beta) \in \mathbb{Z}$.*

It turns out that (abstract) root systems have a very orderly description. Let $\Phi$ be a root system. Then there exists a *polarization* of $\Phi$, which is a splitting $\Phi = \Phi^+ \sqcup \Phi^-$ into the *positive roots* and *negative roots* - which are in fact negatives of each other in $\mathfrak{h}^*$. Furthermore, after choosing a polarization (there are several choices), there exists a **base** $\Delta \subset \Phi^+$ which is unique to each polarization such that each element of $\Phi^+$ can be written uniquely as a $\mathbb{Z}_{\geq 0}$-linear combination of elements of $\Delta$, while each element of $\Phi^-$ can be written uniquely as a $\mathbb{Z}_{\leq 0}$-linear combination of elements of $\Delta$. Furthermore, $\Delta$ forms a basis for $\mathfrak{h}^*$, which follows from the fact that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.

**Example 17.** Let us consider $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, which we can identify as the algebra of $n \times n$ traceless matrices. Then the maximal torus $\mathfrak{t}$ consists of the diagonal matrices. Let $e_i$ denote the linear functional on $\mathfrak{t}$ which takes the $i$th element on the diagonal. Then the roots are precisely $e_i - e_j$ for $i \neq j$. The choice of positive roots can be made to be $\{e_i - e_j \mid i < j\}$, while the negative roots can be chosen to be $\{e_i - e_j \mid i > j\}$. A base can be chose to be $\Delta = \{e_i - e_{i+1} \mid 1 \leq i \leq n-1\}$. Finally, let us consider what each $\mathfrak{g}_\alpha$ is for $\alpha = e_i - e_j$. This is precisely the one-dimensional subspace consisting of matrices $X$ such that $X_{ab} = 0$ for all $(a, b) \neq (i, j)$.

**Example 18.** Specializing Example 17 to $n = 2$, we have the decomposition

$$\mathfrak{sl}_2(\mathbb{C}) \cong \mathbb{C}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Abstractly, we have chosen

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

such that

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f.$$

Note that with an arbitrary complex semisimple Lie algebra $\mathfrak{g}$, for any $\alpha$, we have that $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ are each one dimensional. If we choose $e \in \mathfrak{g}_\alpha$ and $f \in \mathfrak{g}_{-\alpha}$ such that $(e, f) = 2/(\alpha, \alpha)$, then defining $h_\alpha := \alpha^\vee = \frac{2H_\alpha}{(\alpha,\alpha)}$, we have that $(e, f, h_\alpha)$ satisfy the relations

$$[e, f] = h_\alpha, \quad [h_\alpha, e] = 2e, \quad [h_\alpha, f] = -2f.$$

As a result, the subalgebra of $\mathfrak{g}$ generated by $e, f, h_\alpha$ forms a 3-dimensional Lie subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. We call this an **$\mathfrak{sl}_2$-triple**, and denote it by $\mathfrak{sl}_2(\mathbb{C})_\alpha \subset \mathfrak{g}$.

**Definition 19.** Let $E$ be the ambient $\mathbb{R}$-vector space defined by the $\mathbb{R}$-span of $\Delta$. Then to each $\alpha \in \Phi$, we define its **coroot** $\alpha^\vee \in E^*$ by $\langle \alpha^\vee, \lambda \rangle = \frac{2(\alpha,\lambda)}{(\alpha,\alpha)}$. Explicitly, we have $\alpha^\vee = \frac{2H_\alpha}{(\alpha,\alpha)}$ where $H_\alpha \in \mathfrak{h}$ corresponds to $\alpha \in \mathfrak{h}^*$ under the isomorphism given by the Killing form. The set of coroots is denoted $\Phi^\vee$.

**Definition 20.** We now have the **root lattice**, the lattice in $E$ generated by $\alpha \in \Phi$. We also have the **coroot lattice**, the lattice in $E^*$ generated by $\alpha^\vee \in \Phi^\vee$. Note: the coroot lattice is not the dual lattice of the root lattice.

**Definition 21.** A Lie group $G$ is **split** if the maximal torus $T \subset G$ is isomorphic to $(\mathbb{C}^\times)$.

Since $\mathbb{C}$ is algebraically closed, every complex semisimple Lie group is automatically split; furthermore, since semisimple Lie groups are reductive, every complex semisimple Lie group is split reductive as well.

**Definition 22.** The **character lattice** $X^*(G)$ is defined to be $\mathrm{Hom}(T, \mathbb{C}^\times)$. When $G$ is split reductive, then this is equivalent to $\mathrm{Hom}((\mathbb{C}^\times)^n, \mathbb{C}^\times) = \mathbb{Z}^n$.

**Definition 23.** The **cocharacter lattice** $X_*(G)$ is defined to be $\mathrm{Hom}(\mathbb{C}^\times, T)$. When $G$ is split reductive, then this is equivalent to $\mathrm{Hom}(\mathbb{C}^\times, (\mathbb{C}^\times)^n) = \mathbb{Z}^n$.

**Example 24.** Let $G = SL_n(\mathbb{C})$. Then the maximal torus is $\left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} \mid a \in \mathbb{C}^\times\right\} \cong \mathbb{C}^\times$. It follows that the character lattice $X_*(SL_n) \cong \mathbb{Z}$, where $n$ corresponds to the element $\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} \mapsto a^n$. The cocharacter lattice $X^*(SL_n) \cong \mathbb{Z}$, where $n$ corresponds to the element $a \mapsto \begin{pmatrix} a^n & 0 \\ 0 & a^{-n}\end{pmatrix}$.

## 1.2 Our fields of interest

Unlike Lie groups, which are defined over $\mathbb{R}$ or $\mathbb{C}$, the groups we will be studying are defined over non-archimedean local fields. The formal definition is given below, c.f. [All, Def. 0.1]:

**Definition 25.** A **local field** is a field which is locally compact and nondiscrete (there are subsets which are not open).

We will not be too concerned with the precise definition, because there is a complete classification of local fields.

**Theorem 26.** *[All, Thm. 0.6] Every local field is isomorphic (as topological fields) to one of the following:* $\mathbb{R}$, $\mathbb{C}$, *a finite extension of* $\mathbb{Q}_p$ *for some* $p$, *or a finite extension of* $\mathbb{F}_p((T))$ *for some* $p$.

The two more commonly used fields $\mathbb{R}$ and $\mathbb{C}$ are known as *archimedean fields*, while the finite extensions of $\mathbb{Q}_p$ and $\mathbb{F}_p((T))$ are *nonarchimedean fields*. The definition of nonarchimedean fields was given in the first presentation, by the author and M. Haiman.

In this paper, we are primarily focused on characteristic zero non-archimedean local fields, namely finite extensions of $\mathbb{Q}_p$.

**Notation 2.** *From now on,* $K$ *will always denote a finite extension of* $\mathbb{Q}_p$ *for some prime* $p$.

In the case of $K = \mathbb{Q}_p$, we saw that $\mathbb{Q}_p$ is complete with respect to the $p$-adic norm $|\cdot|_p$, $x \mapsto p^{-v_p(x)}$, where $v_p(\cdot)$ is the $p$-adic valuation. Taking all elements where $|x|_p \leq 1$, we have the local ring $\mathbb{Z}_p$ which is compact in $\mathbb{Q}_p$, and $\mathbb{Q}_p = \mathrm{Frac}(\mathbb{Z}_p)$. Furthermore, $\mathbb{Z}_p$ has maximal ideal $p\mathbb{Z}_p$, and has residue field $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z} \cong \mathbb{F}_p$, with $p$ a uniformizer. We can then lift the residue classes of $\mathbb{Z}_p/p\mathbb{Z}_p$ to a set of elements $A \subset \mathbb{Z}_p$ (most commonly the elements $\{0, 1, 2, \ldots, p-1\}$) such that every element of $\mathbb{Z}_p$ can be written as $\sum_{i=0}^\infty a_i p^i$ for $a_i \in A$; it follows that every element of $\mathbb{Q}_p$ can then be written as $\sum_{i \geq N} a_i p^i$ for $a_i \in A$ and some integer $N$.

These results all carry over to finite extensions of $\mathbb{Q}_p$, namely $K$. An equivalent definition of local fields requires that $K$ be complete with respect to a discrete valuation $v$, and hence we may take the (compact) local ring $R \coloneqq \{x \in K \mid v(x) \geq 0\}$ with the property that $K = \mathrm{Frac}(R)$. This local ring has maximal ideal

$\wp \coloneqq \{x \in R \mid v(x) > 0\}$ (note that this automatically includes zero) which is generated by a uniformizer, which we fix to be $\varpi$, such that $\varpi^i R = \wp^i$. The residue field $R/\wp$ is a finite extension of $\mathbb{F}_p$, hence is a finite field $\mathbb{F}_q$. Now we may again lift the elements of $R/\mathfrak{m}$ to a set $A \subset R$ such that every element in $K$ can be written as $\sum_{i \geq N} a_i \varpi^i$ for $a_i \in A$ and $N$ some integer. The topology is the profinite topology c.f. [RZ00], which agrees with the topology on $\mathbb{Q}_p$ as mentioned in presentations from [Gou97] when $K = \mathbb{Q}_p$.

In general, one can assume $K = \mathbb{Q}_p$ for the remainder of this paper, for ease of understanding. Since $\mathbb{Q}_p$ is the easiest to describe explicitly, in examples we will often take $K = \mathbb{Q}_p$, $R = \mathbb{Z}_p$, and $\varpi = p$. However, all results carry over immediately to any finite extension of $\mathbb{Q}_p$ (and most even carry over to any nonarchimedean local field, but we will not discuss that in this paper), by replacing $\mathbb{Z}_p$ with $R$ and $p$ with $\varpi$.

### 1.3 Chevalley groups

What we really want are analogues of Lie groups defined over $K$. The complex semisimple Lie groups are defined over $\mathbb{C}$, but often, as in the case of $SL_n(\mathbb{C})$, there is a fairly obvious way to define $SL_n(k)$ over any field $k$: the group of matrices with entries in $k$ with determinant 1. In [Che55] and later extended in [SFW67], the analogous groups to complex semisimple Lie groups are defined over any field $k$; in our case, we are primarily concerned with $k = K$, a finite extension of $\mathbb{Q}_p$. These groups are called *Chevalley groups*. We will describe the construction below for simply-connected, complex semisimple Lie group.

Let $\widetilde{G}$ be a simply connected, complex semisimple Lie group. Let $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{h}} \oplus \bigoplus_{\gamma \in \Phi} \widetilde{\mathfrak{g}}_\gamma$ be its Lie algebra, with $\widetilde{\mathfrak{h}}$ its Cartan subalgebra. Let $\Phi$ denote its set of roots, and let $\Delta$ be a base (a choice of simple roots). Recall Example 18 for the definition of an $\mathfrak{sl}_2$-triple.

**Theorem 27** (Chevalley)**.** *For each $\gamma \in \Phi$, there exists $X_\gamma \in \widetilde{\mathfrak{g}}_\gamma$ and $H_\gamma \in \widetilde{\mathfrak{h}}$ such that $(X_\gamma, X_{-\gamma}, H_\gamma)$ is an $\mathfrak{sl}_2$-triple for each $\gamma$ and $[H_\gamma, X_\delta] = 2(\delta,\gamma)/(\gamma,\gamma) \cdot X_\delta$ for each $\gamma, \delta \in \Phi$.*

It follows that $\{X_\gamma \mid \gamma \in \Phi\} \cup \{H_\delta \mid \delta \in \Delta\}$ is a basis for $\widetilde{\mathfrak{g}}$.

We now define the corresponding Chevalley group $G$ and its Lie algebra $\mathfrak{g}$.

**Definition 28.** To $\widetilde{G}$, we define the **Chevalley group**

$$G \coloneqq \langle \{\exp(tX_\gamma) \mid t \in K, \gamma \in \Phi\} \rangle.$$

We define its Lie algebra to be

$$\mathfrak{g} \coloneqq \bigoplus_{\delta \in \Delta} K H_\delta \oplus \bigoplus_{\gamma \in \Phi} K X_\gamma.$$

*Remark* 29. The exponential of a matrix is not always well-behaved over fields such as $\mathbb{Q}_p$. However, this can be rectified as in [Rab03] by fixing a certain faithful finite-dimensional representation $\widetilde{V}$ of $\widetilde{G}$, and then taking a full-rank sublattice invariant under all $X_\gamma^n/n!$ and taking the sublattice tensored with $K$ to find a $K$-vector space. We then view each $X_\gamma$ as operators on this $K$-vector space, and by finite-dimensionality, there are a finite set of weights, and hence $X_\gamma$ is nilpotent as an operator. Thus $\exp(tX_\gamma)$ makes sense in this view.

**Example 30.** Let $\widetilde{G} = SL_2(\mathbb{C})$. Its Lie algebra is given by $\mathfrak{sl}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a + d = 0 \right\}$, the traceless $2 \times 2$ matrices. Its Cartan subalgebra consists of $\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$ for $a \in \mathbb{C}$. Let $\lambda_1, \lambda_2$ be the linear functionals on the diagonal matrices, such that $\lambda_1$ picks out the first element on the diagonal, and $\lambda_2$ picks out the second element on the diagonal. The root lattice of $SL_2(\mathbb{C})$ is therefore the sublattice of $\mathbb{Z}\lambda_1 \oplus \mathbb{Z}\lambda_2$ consisting of $\mathbb{Z}(\lambda_1 - \lambda_2)$. There are exactly two roots: $\lambda_1 - \lambda_2$, and $\lambda_2 - \lambda_1$.

Let us now construct the Chevalley group $SL_2(\mathbb{Q}_p)$. Choose $\Delta = \{\lambda_1 - \lambda_2\}$. The subalgebra corresponding to $\lambda_1 - \lambda_2$ is $\mathbb{C}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and the subspace corresponding to $\lambda_2 - \lambda_1$ is $\mathbb{C}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Unsurprisingly (see Example 18), it turns out that setting $X_{\lambda_1-\lambda_2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $X_{\lambda_2-\lambda_1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $H_{\lambda_1-\lambda_2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ gives us an $\mathfrak{sl}_2$-triple. Let $K = \mathbb{Q}_p$, so that we are looking to define $SL_2$ over $\mathbb{Q}_p$. Following Definition 28, we find that

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a, b, c \in \mathbb{Q}_p \right\},$$

taking the $\mathbb{Q}_p$-linear span of the elements in the Chevalley basis, and we have now constructed our Lie algebra over $\mathbb{Q}_p$. We also have

$$G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1, \quad a, b, c, d \in \mathbb{Q}_p \right\},$$

following Definition 28, by exponentiating each $tX_\gamma$. This is exactly what we expect from defining the analogue of $SL_2(\mathbb{C})$ over $\mathbb{Q}_p$: the $2 \times 2$ matrices with entry in $\mathbb{Q}_p$, whose determinant is 1.

One of the most crucial tools for us in this paper will be the $\exp \leftrightarrow d/dt$ dictionary. However, unlike in the complex case, the exponential map does not converge everywhere. In fact, it has a relatively small radius of convergence. Now, when we have an $R$-module $\mathfrak{g}' \subset \mathfrak{g}(K)$ (note that this is **not** a Lie subalgebra), we cannot simply exponentiate everything in $\mathfrak{g}'$ to obtain the corresponding subgroup. However, when $\mathfrak{g}'$ can be written as $\mathfrak{h}' \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}'_\alpha$ for $\mathfrak{h}' \subset \mathfrak{h}$ and $\mathfrak{g}'_\alpha \subset \mathfrak{g}_\alpha$ all $R$-submodules, then we can exponentiate each submodule individually (since $X_\alpha$ is nilpotent for each $\alpha$) and take the subgroup of $G$ generated by these.

**Example 31.** Let $G = SL_2(\mathbb{Q}_p)$ and $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{Q}_p)$. Then consider the $\mathbb{Z}_p$-submodule of $\mathfrak{g}$ given by $\mathfrak{sl}_2(\mathbb{Z}_p)$ - in other words, traceless $2 \times 2$ matrices with entries in $\mathbb{Z}_p$. Since the root decomposition of $\mathfrak{g}$ (as in Theorem 15) is

$$\mathfrak{sl}_2(\mathbb{Q}_p) = \mathbb{Q}_p \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \mathbb{Q}_p \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus \mathbb{Q}_p \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

we note that we can choose a $\mathbb{Z}_p$-module in each summand:

$$\mathfrak{sl}_2(\mathbb{Z}_p) = \mathbb{Z}_p \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \mathbb{Z}_p \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus \mathbb{Z}_p \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

which is closed under $[,]$ and $\mathbb{Z}_p$-action. It follows that the corresponding subgroup of $SL_2(\mathbb{Q}_p)$ is given by exponentiating each individual term and taking the subgroup generated. In other words, we want

$$\begin{aligned}
\left\langle \left\{ \exp\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}, \exp\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}, \exp\begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} \mid t \in \mathbb{Z}_p \right\} \right\rangle &= \left\langle \left\{ \begin{pmatrix} \exp(t) & 0 \\ 0 & 1/\exp(t) \end{pmatrix}, \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \mid t \in \mathbb{Z}_p \right\} \right\rangle, \\
&= \left\langle \left\{ \begin{pmatrix} \exp(t) & 0 \\ 0 & 1/\exp(t) \end{pmatrix}, \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \mid t \in \mathbb{Z}_p \right\} \right\rangle, \\
&= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}_p, \quad ad - bc = 1 \right\}, \\
&= SL_2(\mathbb{Z}_p).
\end{aligned}$$

Notably missing from the list of groups which we would like to work with is $GL_n(K)$, the general linear group. This is a rather serious problem, since $GL_n$ is one of the most fundamental groups in all of mathematics. In fact, almost all of the results of §1.1 and §1.3 carry over to somewhat more general Lie groups called *reductive* Lie groups, and indeed this does include $GL_n$. We refer the reader to [Mil14] for a treatment on algebraic groups, and only point out the differences for reductive Lie groups such as $GL_n$.

One minor difference is that with reductive algebraic groups, we obtain a Lie algebra decomposition $\mathfrak{z} \oplus \mathfrak{g}_{ss}$, where $\mathfrak{z}$ is abelian and $\mathfrak{g}_{ss}$ is semisimple. Therefore, in constructing the Chevalley basis, we must extend a Chevalley basis of $\mathfrak{g}_{ss}$ to include elements of $\mathfrak{z}$.

*Remark* 32. The construction *generally* agrees with our intuition: in most cases, when a (reductive, linear algebraic group) matrix group defined over $\mathbb{C}$ is defined by polynomial equations with integer coefficients, it essentially carries over to any field $K$ verbatim. For example, $GL_n$ over $\mathbb{C}$ is defined by $n \times n$ matrices $(A_{ij})$ over $\mathbb{C}$ such that $\det(A) \neq 0$ (which is a polynomial in the entries). The corresponding $GL_n(K)$ is precisely the same definition: $n \times n$ matrices over $K$ such that the determinant is not zero. In the case of $GL_n$, we construct the linear algebraic group $\operatorname{Spec} \mathbb{Q}\,[x_{11}, \ldots, x_{nn}]_{\det}$ where $\det = \sum_{\sigma \in S_n} \prod_{i=1}^n x_{i\sigma(i)}$ is the determinant. We find that the $\mathbb{C}$-points are exactly $GL_n(\mathbb{C})$, while the $\mathbb{Q}_p$ points are precisely $GL_n(\mathbb{Q}_p)$.

**Example 33.** Recall Example 24. Note that $GL_n(\mathbb{C})$ is split reductive, but not semisimple. However, using the technology of algebraic groups, we can compute its character and cocharacter lattices. Let $G = GL_n(\mathbb{C})$. Then the maximal torus is $\left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{C}^\times \right\} \cong (\mathbb{C}^\times)^2$. It follows that the character lattice $X_*(GL_n) \cong \mathbb{Z}^2$, where $(n, m)$ corresponds to the element $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto a^n b^m$. The cocharacter lattice $X^*(GL_n) \cong \mathbb{Z}^2$, where $(n, m)$ corresponds to the element $a \mapsto \begin{pmatrix} a^n & 0 \\ 0 & a^m \end{pmatrix}$. Note that these are different from the root and coroot lattices of $GL_n(\mathbb{C})$: the (co)character lattice properly contains the (co)root lattice.

# 2 Beginning lattices in $\mathfrak{gl}_n$

Recall that $\mathfrak{gl}_n(K) \cong \mathrm{Mat}_{n\times n}(K)$ as vector spaces, and the Jacobi bracket is given by $[X, Y] \coloneqq XY - YX$. Now we will make use of the topology of $K$. The discrete valuation will give us enough structure to define filtration lattices, which will allow us to define filtration subgroups. Our goal will eventually be to define the Moy-Prasad filtration, but to get a feel for filtration lattices, we will first define two standard lattices in $\mathfrak{gl}_n(K)$, following [DeB, § 2.2] and [DeB04, § 2.3].

**Definition 34.** A **lattice** in a Lie algebra $\mathfrak{g}$ is a compact open submodule of $\mathfrak{g}$.

Using the $\exp \leftrightarrow d/dt$ dictionary to move from $\mathfrak{g} \leftrightarrow G$, we see that filtrations of $\mathfrak{g}$ give rise to filtrations of $G$ by compact open subgroups.

## 2.1 Congruence filtration lattices

The first system of lattices are the **congruence filtration lattices**, defined for nonnegative integers $i \geq 0$ by

$$\mathfrak{k}_i \coloneqq \mathrm{Mat}_{n\times n}(\varpi^i R).$$

**Example 35.** Let $n = 3$ and $K = \mathbb{Q}_p$, so that $R = \mathbb{Z}_p$ and $\wp = p\mathbb{Z}_p$. Then the congruence filtration lattices of

$\mathfrak{gl}_3(\mathbb{Q}_p)$ are given by

$$\mathfrak{k}_0 = \begin{pmatrix} R & R & R \\ R & R & R \\ R & R & R \end{pmatrix} \supset \mathfrak{k}_1 = \begin{pmatrix} \wp & \wp & \wp \\ \wp & \wp & \wp \\ \wp & \wp & \wp \end{pmatrix} \supset \mathfrak{k}_2 = \begin{pmatrix} \wp^2 & \wp^2 & \wp^2 \\ \wp^2 & \wp^2 & \wp^2 \\ \wp^2 & \wp^2 & \wp^2 \end{pmatrix} \supset \mathfrak{k}_3 = \begin{pmatrix} \wp^3 & \wp^3 & \wp^3 \\ \wp^3 & \wp^3 & \wp^3 \\ \wp^3 & \wp^3 & \wp^3 \end{pmatrix} \dots$$

Note that $\mathfrak{k}_0/\mathfrak{k}_1 \cong \mathfrak{gl}_n(\mathbb{F}_q)$.

Using the $\exp \leftrightarrow d/dt$ dictionary to move between $\mathfrak{gl}_n(K)$ and $GL_n(K)$, we consider the corresponding subgroups in $GL_n(K)$. For $i \geq 0$, define the **standard filtration subgroups** by

$$K_i \coloneqq \langle \exp tX \mid t \in R, X \in \mathfrak{k}_i \rangle = \begin{cases} \mathfrak{k}_0^\times & i = 0, \\ 1 + \mathfrak{k}_i & i > 0. \end{cases}$$

Note that this indeed produces a filtration

$$K_0 \triangleright K_1 \triangleright K_2 \triangleright K_3 \triangleright \cdots$$

such that each $K_{i+1}$ is normal in $K_i$; furthermore, the $K_i$ are indeed compact open subgroups. In fact, $K_0$ is the maximal compact open subgroup of $GL_n(\mathbb{Q}_p)$ up to conjugation.

## 2.2 Standard Iwahori filtration

The second system of lattices are the **standard Iwahori filtration lattices**, defined for rational numbers $m/n$ for nonnegative integers $m \geq 0$ by

$$\mathfrak{b}_{m/n} \coloneqq \{X \in \mathrm{Mat}_{n\times n}(R) \mid X_{ij} \in \varpi^{\lceil \frac{m+i-j}{n} \rceil} R\}.$$

**Example 36.** Let $n = 3$ and $K = \mathbb{Q}_p$, so that $R = \mathbb{Z}_p$ and $\wp = p\mathbb{Z}_p$. Then the standard Iwahori filtration lattices of $\mathfrak{gl}_3(\mathbb{Q}_p)$ are given by

$$\mathfrak{b}_0 = \begin{pmatrix} R & R & R \\ \wp & R & R \\ \wp & \wp & R \end{pmatrix} \supset \mathfrak{b}_{1/3} = \begin{pmatrix} \wp & R & R \\ \wp & \wp & R \\ \wp & \wp & \wp \end{pmatrix} \supset \mathfrak{b}_{2/3} = \begin{pmatrix} \wp & \wp & R \\ \wp & \wp & \wp \\ \wp^2 & \wp & \wp \end{pmatrix} \supset \mathfrak{b}_1 = \varpi\mathfrak{b}_0 = \begin{pmatrix} \wp & \wp & \wp \\ \wp^2 & \wp & \wp \\ \wp^2 & \wp^2 & \wp \end{pmatrix} \dots$$

Once again, we use the $\exp \leftrightarrow d/dt$ dictionary to obtain a filtration of $GL_n(\mathbb{Q}_p)$ by compact open subgroups $B_{m/n}$, defined by

$$B_{m/n} \coloneqq \langle \exp tX \mid t \in R, X \in \mathfrak{b}_{m/n} \rangle = \begin{cases} \mathfrak{b}_0^\times & m = 0, \\ 1 + \mathfrak{b}_{m/n} & m > 0. \end{cases}$$

As in the case of the standard filtration subgroups, we have that the $B_t$ form a filtration of compact open subgroups $B_0 \triangleright B_{1/n} \triangleright B_{2/n} \triangleright B_{3/n} \triangleright B_{4/n} \triangleright \cdots$ where each $B_{(i+1)/n}$ is normal in $B_{i/n}$.

# 3 Bruhat-Tits apartment

Now, let us fix a complex semisimple Lie group $\widetilde{G}$ with Lie algebra $\widetilde{\mathfrak{g}}$. Following §1.3, we let $G$ be the corresponding Chevalley group over $K$, and $\mathfrak{g}$ its Lie algebra over $K$. We fix a Chevalley basis $\{H_\gamma, X_\gamma, X_{-\gamma} \mid \gamma \in \Phi^+\}$ and $\Delta$ a base. Recall that the coroot lattice is $\mathbb{Z}\{\Phi^\vee(G)\}$. We now define the apartment of $G$.

**Definition 37.** The **apartment** of $G$, denoted by $\mathcal{A}(G)$ (or shorthand by $\mathcal{A}$), is defined to be

$$\mathcal{A} \coloneqq X_*(G) \otimes \mathbb{R}/(X_*(Z(G)) \otimes \mathbb{R}),$$

where $X_*(G)$ is the cocharacter lattice.

Since the coroot lattice lies inside the cocharacter lattice, after tensor with $\mathbb{R}$, they may be canonically identified. Therefore we have an equivalent definition instead, using the coroot lattice:

*Remark* 38. The apartment of $G$ is canonically identified with

$$\mathcal{A} \cong \mathbb{Z}\Phi^\vee \otimes \mathbb{R}.$$

**Example 39.** When $G = GL_n$, the roots are given by $\alpha_{ij} \coloneqq e_i - e_j$ for $i \neq j$, where $e_i$ is the linear functional on $\mathfrak{t}$ sending $\mathrm{diag}(t_1, \dots, t_n) \mapsto t_i$. The coroots are given by $\lambda_{ij} \coloneqq e^i - e^j$ for $i \neq j$, where $e^i(t) = \mathrm{diag}(0, 0, \dots, 0, t, 0, \dots, 0)$ (with a $t$ at the $i$th index and zero everywhere else). Therefore

$$\mathcal{A}(GL_n) \cong \bigoplus_{i=1}^{n-1} \mathbb{R}\{e^i - e^j\} \cong \bigoplus_{i=1}^{n} \mathbb{R}e^i / \left(\mathbb{R}\sum_{i=1}^{n} e^i\right).$$

Alternatively, we can do this via character and cocharacter lattices as well. The characters of $GL_n$ are precisely $\alpha_i$ for $i = 1, 2, \dots, n$, which act on $T$ (the diagonal matrices in $GL_n$) by picking out the $i$th element on the diagonal. The cocharacters of $GL_n$ are $\lambda_i$ for $i = 1, 2, \dots, n$, which act on $K$ by sending $t \mapsto \mathrm{diag}(1, \dots, 1, t, 1, \dots, 1)$. Since $Z(GL_n)$ consists of the multiples of the identity matrix, it follows that $X_*(Z(G)) \cong \mathbb{Z}\lambda \subset \bigoplus_{i=1}^n \mathbb{Z}\lambda_i$ where $\lambda = \lambda_1 + \cdots + \lambda_n$. Therefore

$$\mathcal{A}(GL_n) \cong \bigoplus_{i=1}^{n} \mathbb{R}\lambda_i / \left(\mathbb{R}\sum_{i=1}^{n} \lambda_i\right).$$

It is easy to see that these are equivalent.

The apartment of $G$ has a natural action by $\Phi$, since $\Phi$ acts naturally on $\Phi^\vee$ (note that $\Phi^\vee \subset X_*(G)$, so this still makes sense in the first construction). Now define

$$\Psi \coloneqq \{\alpha + n \mid \alpha \in \Phi, n \in \mathbb{Z}\}$$

such that

$$(\alpha + n)\left(\sum \lambda_i \otimes r_i\right) = n + \sum r_i \langle \lambda_i, \alpha \rangle,$$

hence each element of $\Psi$ determines a function $\mathcal{A} \to \mathbb{R}$.

For each $\psi \in \Psi$, define hyperplanes

$$H_\psi \coloneqq \{x \in \mathcal{A} \mid \psi(x) = 0\}.$$

These hyperplanes provide a simplicial decomposition of $\mathcal{A}$, as illustrated in the next two examples taken from [DeB04].

**Example 40.** Let $G = GL_2$. The cocharacters are $\lambda_1$ and $\lambda_2$, which act by $\lambda_1(s)\lambda_2(t) = \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix}$. The center is generated by $\lambda_1(s)\lambda_2(s)$, hence $X_*(Z(G)) \cong \mathbb{Z}(\lambda_1 + \lambda_2)$. Thus we have that $\mathcal{A} \cong \mathbb{R}\lambda_1 \oplus \mathbb{R}\lambda_2/(\lambda_1 + \lambda_2)$, hence $\mathcal{A}$ is one-dimensional. Fixing $x_0$ as the basepoint, we have the illustration of $\mathcal{A}(GL_2(K))$ in Figure 1.

**Example 41.** Let $G = GL_3$. The cocharacters are $\lambda_1$, $\lambda_2$, and $\lambda_3$, which act by $\lambda_1(r)\lambda_2(s)\lambda_3(t) = \begin{pmatrix} r & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & t \end{pmatrix}$. The center is generated by $\lambda_1(s)\lambda_2(s)\lambda_3(s)$, hence $X_*(Z(G)) \cong \mathbb{Z}(\lambda_1 + \lambda_2 + \lambda_3)$. Thus we have that $\mathcal{A} \cong$

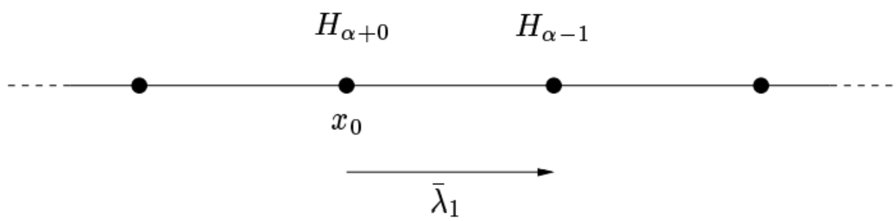


Figure 1: Apartment for $GL_2(K)$. Figure taken from [DeB04].

$\mathbb{R}\lambda_1 \oplus \mathbb{R}\lambda_2 \oplus \mathbb{R}\lambda_3/(\lambda_1 + \lambda_2 + \lambda_3)$, hence $\mathcal{A}$ is two-dimensional. Fixing $x_0$ as the basepoint, we have the illustration of $\mathcal{A}(GL_3(K))$ in Figure 2.

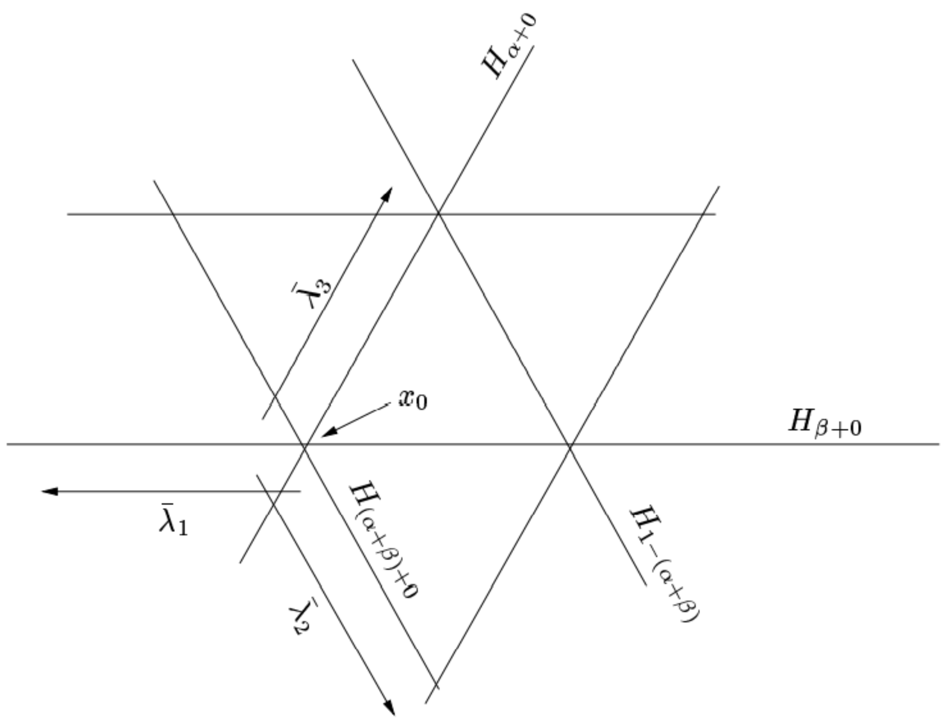


Figure 2: Apartment for $GL_3(K)$. Figure taken from [DeB04].

# 4 Moy-Prasad filtrations

First, we define a filtration of the split torus $T$. Recall that a split torus is isomorphic to a product of $\mathbb{G}_m$, and hence the $K$-points are isomorphic to a product of copies of $K^\times$. Therefore, we will use the coordinates $(t_1, \ldots, t_n)$ to denote elements of $T$.

**Definition 42.** For all nonnegative integers $i$, let $\mathfrak{t}_i := \mathfrak{t}(\wp^i)$; the corresponding subgroup in $T$ is therefore $T_i := T(1+\wp^i)$, or more explicitly, $\{(t_1, \dots, t_n) \mid t_j \in 1 + \varpi^i R \text{ for all } j\}$.

Notice that this indeed gives a filtration

$$\mathfrak{t}_0 \supset \mathfrak{t}_1 \supset \mathfrak{t}_2 \supset \dots$$

in much the same way as the congruence filtration lattices; the same analogy applies to the $T_i$. However, note that the apartment $\mathcal{A}(G)$ is a real vector space. We therefore extend the definition to all $r \in \mathbb{R}_{\geq 0}$ as follows.

**Definition 43.** For any real number $r \geq 0$, define $\mathfrak{t}_r := \mathfrak{t}_{\lceil r \rceil}$. Similarly, we define $T_r := T_{\lceil r \rceil}$.

**Example 44.** When $G = GL_3$, we have the following filtrations:

$$\mathfrak{t}_0 = \begin{pmatrix} R & & \\ & R & \\ & & R \end{pmatrix} \supset \mathfrak{t}_1 = \begin{pmatrix} \wp & & \\ & \wp & \\ & & \wp \end{pmatrix} \supset \mathfrak{t}_2 = \begin{pmatrix} \wp^2 & & \\ & \wp^2 & \\ & & \wp^2 \end{pmatrix} \supset \mathfrak{t}_3 = \begin{pmatrix} \wp^3 & & \\ & \wp^3 & \\ & & \wp^3 \end{pmatrix} \supset \cdots$$

with the corresponding subgroups given by

$$T_0 = \begin{pmatrix} R^\times & & \\ & R^\times & \\ & & R^\times \end{pmatrix} \supset T_1 = \begin{pmatrix} 1+\wp & & \\ & 1+\wp & \\ & & 1+\wp \end{pmatrix} \supset T_2 = \begin{pmatrix} 1+\wp^2 & & \\ & 1+\wp^2 & \\ & & 1+\wp^2 \end{pmatrix} \supset \cdots$$

Now consider that the Lie algebra $\mathfrak{g}$ decomposes into a direct sum (as $\mathfrak{t}$-representations) $\mathfrak{z} \oplus \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$. Each $\mathfrak{g}_\alpha$ is one dimensional, and can be identified with $\mathfrak{g}_a(K)$, the Lie algebra of $\mathbb{G}_a$, which is in turn canonically identified with $K$. Therefore we have a natural system of filtrations of $\mathfrak{g}_\alpha$ indexed by the integers, namely $\wp^n X_\alpha$ (with $X_\alpha$ the element of the Chevalley basis corresponding to $\alpha$).

**Definition 45.** For any $\psi \in \Psi$, let $\psi = \alpha + n$ and define $\mathfrak{g}_\psi$ as the $R$-module $\{tX_\alpha \mid t \in \wp^n\}$.

Putting these together, we can define the Moy-Prasad filtrations, following [DeB04, §3.4].

**Definition 46** (Moy-Prasad filtration)**.** For any $x \in \mathcal{A}$ and real number $r \geq 0$, define

$$\mathfrak{g}_{x,r} := \mathfrak{t}_r \oplus \sum_{\psi \in \Psi \mid \psi(x) \geq r} \mathfrak{g}_\psi = \mathfrak{t}_r \oplus \sum_{\alpha \in \Phi} \sum_{m \in \mathbb{Z} \mid \alpha(x)+m \geq r} \mathfrak{g}_{\alpha+m},$$
$$\mathfrak{g}_{x,r^+} := \mathfrak{t}_r \oplus \sum_{\psi \in \Psi \mid \psi(x) > r} \mathfrak{g}_\psi = \mathfrak{t}_r \oplus \sum_{\alpha \in \Phi} \sum_{m \in \mathbb{Z} \mid \alpha(x)+m > r} \mathfrak{g}_{\alpha+m}.$$

Immediately, we find the analogous subgroups of $G$:

**Definition 47** (Moy-Prasad filtration of subgroups)**.** For any $x \in \mathcal{A}$ and real number $r \geq 0$, define

$$G_{x,r} := \langle T_r, \{\exp(tX_\alpha) \mid \alpha \in \Phi, t \in \wp^{-\lfloor \alpha(x) - r \rfloor}\}\rangle,$$
$$G_{x,r^+} := \langle T_{\lfloor r \rfloor + 1}, \{\exp(tX_\alpha) \mid \alpha \in \Phi, t \in \wp^{1-\lfloor \alpha(x) - r \rfloor}\}\rangle.$$

It is not immediate how the $G_{x,r}$ and $G_{x,r^+}$ form a filtration, but the following results taken from [Rab03, Remark 5.3] give some insight.

**Proposition 48.** *[Rab03]*

- *For $r > s$, we have $G_{x,r} \subset G_{x,s}$.*
- *For all $g \in G(K)$, we have $G_{gx,r} = gG_{x,r}g^{-1}$ and $G_{gx,r^+} = gG_{x,r^+}g^{-1}$.*

- *We have $G_{x,r^+} \triangleleft G_{x,r}$, and $G_{x,r}/G_{x,r^+}$ is always finite. If $r > 0$, the quotient is abelian as well.*
- *For any $x$ and $r$, there exists some $\epsilon \geq 0$ such that $G_{x,r^+} = G_{x,r+\epsilon}$.*

**Example 49** (Moy-Prasad filtrations for $GL_2(K)$)**.** Recall the apartment for $GL_2(K)$ from Example 40. Let us consider first the $G_{x,r}$ for $r = 0$, corresponding to $x = x_0 + t\overline{\lambda_1}$ for $t \in \mathbb{R}$. Since $r = 0$, we have $T_r = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$, the diagonal matrices in $GL_2(K)$. The two roots are $\alpha_1$ and $\alpha_2$, corresponding to $e^1 - e^2$ and $e^2 - e^1$, respectively. We find a Chevalley basis given by

$$\begin{aligned} X_{\alpha_1} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ H_{\alpha_1} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ X_{\alpha_2} &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\ Z &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

On the other hand, the cocharacters are $\lambda_1$ and $\lambda_2$, as described in Example 40. Since $\langle \lambda_i, \alpha_j \rangle = (-1)^{i+j}$, we have that $\alpha_1(x) = t$ and $\alpha_2(x) = -t$.

Suppose $t = 0$. In accordance to the $\begin{pmatrix} R & R \\ R & R \end{pmatrix}$ sublattice of $\mathfrak{gl}_2(K)$ lying directly over $x_0$ in the diagram, we have

$$\begin{aligned} G_{x_0,0} &= \langle T(R), \{\exp(sX_{\alpha_1}), \exp(sX_{\alpha_2}) \mid s \in R\} \rangle, \\ &= \left\langle \left\{ \begin{pmatrix} s & 0 \\ 0 & r \end{pmatrix}, \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} \mid s, r \in R \right\} \right\rangle, \\ &= GL_2(R). \end{aligned}$$

Suppose $t = 1$. In accordance to the $\begin{pmatrix} R & \wp^{-1} \\ \wp & R \end{pmatrix}$ sublattice of $\mathfrak{gl}_2(K)$ lying directly over $x_0 + \overline{\lambda_1}$ in the diagram, we have

$$\begin{aligned} G_{x_0+\overline{\lambda_1},0} &= \langle T(R), \{\exp(sX_{\alpha_1}) \mid s \in \wp^{-1}\}, \{\exp(sX_{\alpha_2}) \mid s \in \wp\} \rangle, \\ &= \left\langle \left\{ \begin{pmatrix} s & 0 \\ 0 & r \end{pmatrix} \mid s, r \in R \right\}, \left\{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \mid s \in \wp^{-1} \right\}, \left\{ \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} \mid r \in \wp \right\} \right\rangle, \\ &= \begin{pmatrix} R & \wp^{-1} \\ \wp & R \end{pmatrix} \cap GL_2(K). \end{aligned}$$

Let us now calculate $G_{x,r}$ for $x \in \{x_0, x_0 + \overline{\lambda_1}\}$ and $r = 1$. We will see that these match the subgroups corresponding to the sublattices $\begin{pmatrix} \wp & \wp \\ \wp & \wp \end{pmatrix}$ and $\begin{pmatrix} \wp & R \\ \wp^2 & \wp \end{pmatrix}$ as in the diagram. We find that $G_{x_0,1}$ corresponds to the corresponding subgroup in $GL_2(K)$ of $\begin{pmatrix} \wp & \wp \\ \wp & \wp \end{pmatrix}$:

$$\begin{aligned} G_{x_0,1} &= \langle T(\wp), \{\exp(sX_{\alpha_1}), \exp(sX_{\alpha_2}) \mid s \in \wp\} \rangle, \\ &= \left\langle \left\{ \begin{pmatrix} 1+s & 0 \\ 0 & 1+r \end{pmatrix}, \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} \mid s, r \in \wp \right\} \right\rangle, \end{aligned}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \wp & \wp \\ \wp & \wp \end{pmatrix}.$$

Similarly, we find that $G_{x_0+\overline{\lambda_1},1}$ corresponds to the corresponding subgroup in $GL_2(K)$ of $\begin{pmatrix} \wp & R \\ \wp^2 & \wp \end{pmatrix}$:

$$\begin{aligned} G_{x_0+\overline{\lambda_1},1} &= \langle T(\wp), \{\exp(sX_{\alpha_1}) \mid s \in R\}, \{\exp(sX_{\alpha_2}) \mid s \in \wp^2\}\rangle, \\ &= \left\langle \left\{ \begin{pmatrix} 1+s & 0 \\ 0 & 1+r \end{pmatrix} \mid s, r \in \wp \right\}, \left\{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \mid s \in R \right\}, \left\{ \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} \mid r \in \wp^2 \right\} \right\rangle, \\ &= \begin{pmatrix} 1+\wp & R \\ \wp^2 & 1+\wp \end{pmatrix} \cap GL_2(K). \end{aligned}$$

More Moy-Prasad filtration lattices in $\mathfrak{gl}_2(K)$ can be seen in Figure 3.

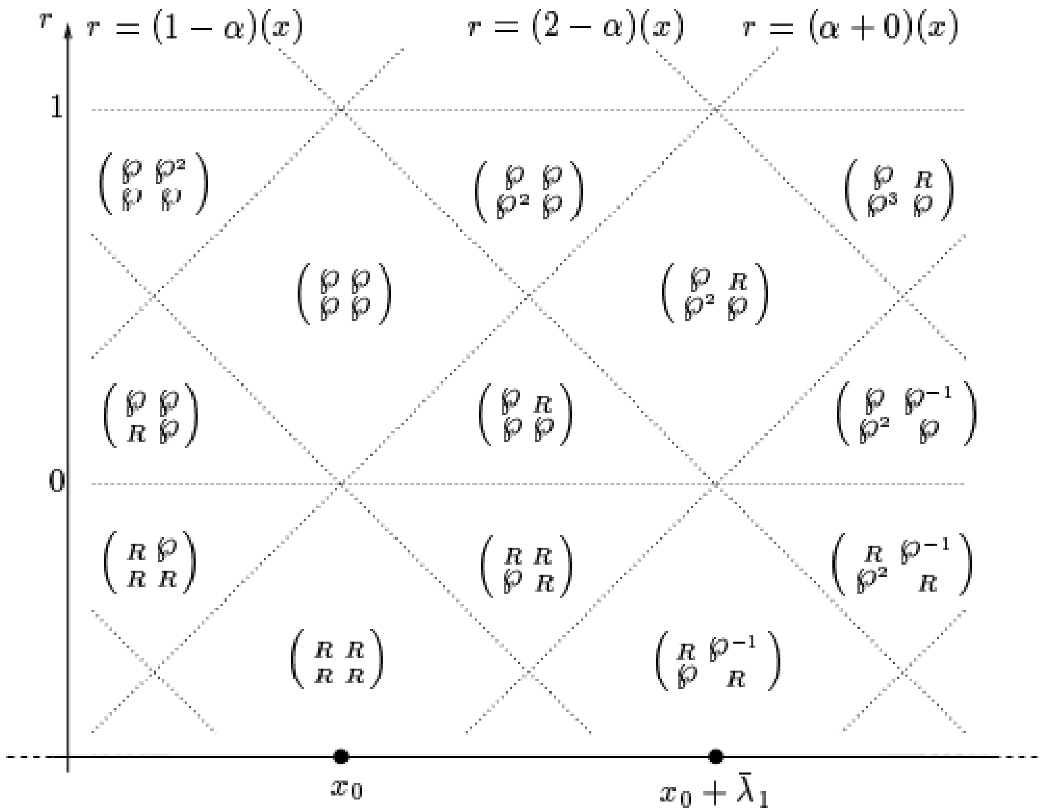


Figure 3: Diagram of Moy-Prasad filtrations for $\mathfrak{gl}_2(K)$, taken from [DeB04].

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Zawad Chowdhury feedback: 10/10
Rachana Madhukara feedback: 10/10